VŠB – Technical University of Ostrava
Faculty of Mechanical Engineering
Department of Control Systems and Instrumentation
Czech Republic

STATE SPACE CONTROL

Prof. Ing. Miluše Vítečková, CSc.
Prof. Ing. Antonín Víteček, CSc., Dr.h.c.

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Reviewer: Doc. Ing. Milan Heger, CSc.

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PREFACE

The textbook "State Space Control" is devoted to the fundamentals of the automatic control. The main emphasis is put on the description of the principles of the state space models and the negative feedback, and their use for the linear dynamic system control. It deals with the most important area of the state space control of the SISO systems.

Since the textbook discuses only fundamentals of the state space control, in the text are not given accurate proofs. For a deeper and broader study, the following publications are recommended:

OGATA, K. *Modern Control Engineering.* 5th Edition. Prentice-Hall, Boston, 2010

FRANKLIN, G.F., POWELL, J.D. , EMAMI-NAEINI, A. *Feedback Control of Dynamic Systems.* 4th Edition. Prentice-Hall, Upper Saddle River, New Jersey, 2002

MANDAL, A. K. *Introduction to Control Engineering. Modeling, Analysis and Design*. New Age Internationsl (P) Publishers, New Delhi, 2006

NISE, N. S. *Control Systems Engineering*. 6th Edition. John Wiley and Sons, Hoboken, New Jersey, 2011

NOSKIEVIČ, P. *Modelling and Simulation of Mechatronic Systems using MATLAB-Simulink.* VŠB-TU Ostrava, 2013

It is assumed that students have basic knowledge of the classical automatic control in the range of textbook, e.g.:

VÍTEČEK, A., VÍTEČKOVÁ, M. *Closed-loop Control of Mechatronic Systems.* VŠB-TU Ostrava, 2013

The textbook is determined for students who are interested in the automatic control theory.

CONTENTS

LIST OF BASIC NOTATIONS AND SYMBOLS

$a, a_i, b, b_i, \ldots$ constants

a_i coefficients of left side of differential equation, coefficients of transfer function denominator

a_i^l desired characteristic polynomial coefficients of observer

$\boldsymbol{a}^l$ vector of desired characteristic polynomial coefficients of observer

a_i^w desired characteristic polynomial coefficients of closed-loop control system

$\boldsymbol{a}^w$ vector of desired characteristic polynomial coefficients of closed-loop control system

$A(\omega) = \mathrm{mod}G(\mathrm{j}\omega) = |G(\mathrm{j}\omega)|$ frequency transfer function modulus, plot of $A(\omega)$ = magnitude response

$\boldsymbol{A}$ system (dynamics) matrix of order n $[(n\times n)]$

$\boldsymbol{A}_w$ system matrix of closed-loop control system of order n $[(n\times n)]$

$\boldsymbol{A}_l$ system matrix of observer of order n $[(n\times n)]$

b_i coefficients of right side of differential equation, coefficients of transfer function nominator

$\boldsymbol{b}$ input state vector of dimension n

$\boldsymbol{c}$ output state vector of dimension n

C capacitance

d transfer constant

e control error

$e(\infty)$ steady-state error

f general function

$f = \dfrac{\omega}{2\pi}$ frequency

$g(t)$ impulse response

$G(s)$ transfer function, transform of impulse response

$G(\mathrm{j}\omega) = P(\omega) + \mathrm{j}Q(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\varphi(\omega)}$ frequency transfer function, plot of $G(j\omega)$ = frequency response

$h(t)$ step response

$H(s)$ transform of step response

i current

$\mathrm{j} = \sqrt{-1}$ imaginary unit

k relative discrete time ($k = 0,1,2,\ldots$)

k_i	gain
k_w	coefficient of input filter, input correction
kT	discrete time
K_I	weight of controller integral component (term)
K_P	controller gain, weight of controller proportional component (term)
$\boldsymbol{k}$	vector of state space controller
L	inductance
L	operator of direct Laplace transform
L^{-1}	operator of inverse Laplace transform
$L(\omega) = 20\log A(\omega)$	logarithmic modulus of frequency transfer function
$\boldsymbol{l}$	Luenberger observer gain vector, correction vector
m	degree of polynomial in transfer function nominator, motor torque, mass
m_l	load torque
$m_L = 20\log m_A$	logarithmic gain margin
M	polynomial in transfer function nominator (roots = zeros)
n	degree of characteristic polynomial, degree of polynomial in transfer function denominator, dimension of state variable vector $\boldsymbol{x}$
N	characteristic polynomial or quasipolynomial, polynomial or quasipolynomial in transfer function denominator (roots = poles)
N_k	characteristic polynomial of closed-loop control system with state controller
N_{kw}	desired characteristic polynomial of closed-loop control system with state controller
N_l	characteristic polynomial of observer
N_{lw}	desired characteristic polynomial of observer
$P(\omega) = \mathrm{Re}G(\mathrm{j}\omega)$	real part of frequency transfer function
p_i	poles of observer
$Q(\omega) = \mathrm{Im}G(\mathrm{j}\omega)$	imaginary part of frequency transfer function
$\boldsymbol{Q}_{co}$	controllability matrix of order n [$(n\times n)$]
$\boldsymbol{Q}_{ob}$	observability matrix of order n [$(n\times n)$]
R	resistance
$s = \alpha + \mathrm{j}\omega$	complex variable, independent variable in Laplace transform
s_i	poles of linear dynamic system = roots of polynomial $N(s)$
s_j^0	zeros of linear dynamic system = roots of polynomial $M(s)$
s_i^w	desired poles of closed-loop control system with state controller
t	(continuous) time

t_s settling time

$t_\varphi = \dfrac{\varphi}{\omega}$ time corresponding to phase φ

$T = \dfrac{2\pi}{\omega}$ period

T sampling period, period

T_d time delay (dead time)

T_D derivative time

T_I integral time

T_i (inertial) time constant

$\boldsymbol{T}_c$, $\boldsymbol{T}_o$ transformation matrices of order n [$(n \times n)$]

u manipulated variable, control variable, input variable (input), voltage

u_T formed (stair case) manipulated variable

v disturbance variable (disturbance)

w desired (reference, command) variable, set-point value

x state variable (state)

$\boldsymbol{x}$ state vector (state) of dimension n

y controlled (plant, process) variable, output variable (output)

y_w response caused by desired variable

y_T transient part of response

y_S steady-state part of response

Z impedance

α stability degree (absolute damping)

$\alpha = \mathrm{Re}\, s$ real part of the complex variable s

$\delta(t)$ unit Dirac impulse

Δ difference

$\boldsymbol{\varepsilon}$ state error vector

$\eta(t)$ unit Heaviside step

$\omega = 2\pi f$ angular frequency, angular speed

$\omega = \mathrm{Im}\, s$ imaginary part of complex variable s

ω_0 natural angular frequency

$\varphi(\omega) = \arg G(\mathrm{j}\omega)$ phase of frequency transfer function, plot of $\varphi(\omega)$ = phase response

ξ_i relative damping

κ overshoot

τ_j time constant

Upper indices

*	recommended, optimal
-1	inverse
T	transpose

Lower indices

c	controller, control
co	controllability
d	diagonal
D	discrete
o	observer, observation
ob	observability
w	desired
t	transformed, transformation

Symbols over letters

$\cdot$	(total) derivative with respect to time
$\wedge$	estimation

Relation signs

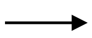
$\approx$	approximately equal
$\doteq$	after rounding equal
$\widehat{=}$	correspondence between original and transform
$\Rightarrow$	implication
$\Leftrightarrow$	equivalence

Graphic marks

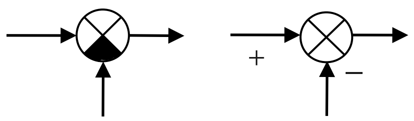
○	single zero
◎	double zero
×	single pole
⋇	double pole


nonlinear system (element)


linear system (element)


single variable (signal)


multiple variable (signal)



summing node (filled segment expresses minus sign)

Shortcuts

arg	argument
dB	decibel
const	constant
dec	decade
det	determinant
dim	dimension
Im	imaginary, imaginary part
lim	limit
max	maximum
min	minimum
mod	modulus
Re	real, real part
sign	signum

1 INTRODUCTION

Conventional controllers P, I, PD, PI and PID have a simple structure and when appropriately tuned they are able to ensure for common controlled systems (plants) a relatively good quality of control processes. Their advantage is a low cost, an easy implementation and a simple tuning that do not require deep theoretical knowledge. A properly designed and tuned conventional controller is able to ensure both following of changes in the desired variable, and enough suppressing negative influence of disturbances. A conventional control is also robust because it is able to ensure the required control quality for given changes in properties of controlled systems.

In some cases the use of conventional controllers cannot guarantee the required control quality. It is especially in the case of unstable and complex controlled systems and for high requirements for control quality. In this case it is advisable to use state space control. Its birth and development is associated with an aeronautics and astronautics. In the state space control theory the general concepts of the system theory are used.

The state space control removes some disadvantages of the conventional control, it allows significantly increase the control quality, but it requires some theoretical knowledge.

In textbook, there are only given basic approaches and methods used in the analysis and synthesis of SISO continuous feedback control systems in the state space. The text is arranged in a way, that allows easy extension to discrete and MIMO control systems.

2 MATHEMATICAL MODELS OF DYNAMIC SYSTEMS

2.1 General mathematical models

For the design and study of the properties of systems we use their **mathematical models**. It is very advantageous because experimentation with real systems may be substituted by experimentation with their mathematical models, i.e. by **simulation**. It enables considerable reductions in cost and risk of damage to the real system. It is also important for accelerating the whole process. New nontraditional solutions often arise.

In automatic control theory in the time domain, mathematical models have forms which are algebraic, transcendental, differential, partial differential, integral, difference, summation equations and their combinations. The mathematical model can be obtained by **identification** using an analytical or experimental method, if necessary by a combination of them. For example, a mathematical model can be obtained analytically and its parameters can be refined experimentally. Sometimes term identification means finding a mathematical model using an experimental method. We will only deal with such mathematical models that can be expressed in the forms of the *t*-invariant (stationary) ordinary differential equations, which describe real systems with lumped parameters.

When evaluating a mathematical model and the simulation results we must always remember that ***every mathematical model is only an approximation of the real system***.

Since even a very complex MIMO (**m**ulti-**i**nput **m**ulti-**o**utput) system is formed by combining SISO (**s**ingle-**i**nput **s**ingle-**o**utput) systems, main attention will be paid to SISO systems.

Consider the **SISO system** which is described by the generally nonlinear differential equation

$$g[y^{(n)}(t),\dots,\dot{y}(t),y(t),u^{(m)}(t),\dots,\dot{u}(t),u(t)]=0\,. \tag{2.1a}$$

$$\left.\begin{aligned} &\dot{y}(t)=\frac{\mathrm{d}y(t)}{\mathrm{d}t},\ y^{(i)}(t)=\frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i};\ \ i=2,3,\dots,n,\\ &\dot{u}(t)=\frac{\mathrm{d}u(t)}{\mathrm{d}t},\ u^{(j)}(t)=\frac{\mathrm{d}^j u(t)}{\mathrm{d}t^j};\ \ j=2,3,\dots,m, \end{aligned}\right\} \tag{2.1b}$$

with initial conditions

$$\left.\begin{aligned} &y(0)=y_0,\dot{y}(0)=\dot{y}_0,\dots,y^{(n-1)}(0)=y_0^{(n-1)},\\ &u(0)=u_0,\dot{u}(0)=\dot{u}_0,\dots,u^{(m-1)}(0)=u_0^{(m-1)}, \end{aligned}\right\} \tag{2.1c}$$

where $u(t)$ is the **input variable** (signal) = **input**, $y(t)$ – the **output variable** (signal) = **output**, g – the generally nonlinear function, n – the **system order**.

If the inequality

$$n > m \tag{2.2}$$

holds, then the mathematical model satisfies a **strong physical realizability condition**.

In case

$$n = m \tag{2.3}$$

it satisfies only a **weak** physical realizability condition.

For

$$n < m \tag{2.4}$$

the mathematical model is **not physically realizable** and therefore it does not express the behaviour of the real system.

The mathematical model (2.1a), in which the derivatives appear (2.1b), describes the **dynamic** (dynamical) **system** (it has a memory).

From the differential equation (2.1a) for

$$\lim_{t\to\infty} y^{(i)}(t) = 0; \; i = 1,2,\ldots,n,$$
$$\lim_{t\to\infty} u^{(j)}(t) = 0; \; j = 1,2,\ldots,m$$

it is possible to obtain the equation (if it exists)

$$y = f(u), \tag{2.5}$$

where

$$\left.\begin{aligned} y &= \lim_{t\to\infty} y(t), \\ u &= \lim_{t\to\infty} u(t). \end{aligned}\right\} \tag{2.6}$$

The equation (2.5) expresses the **static characteristic** of the given dynamic system (2.1), see e.g. Fig. 2.1.

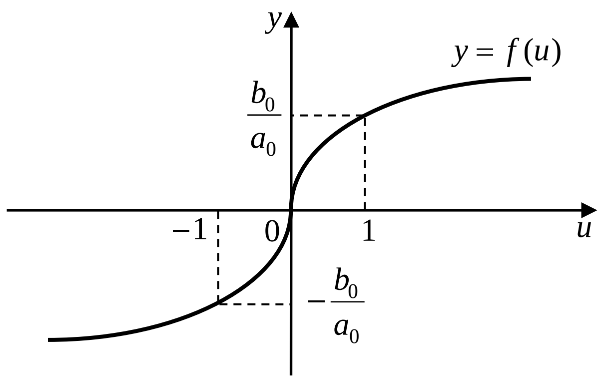


Fig. 2.1 Nonlinear static characteristic – Example 2.1

A static characteristic describes the dependency between output y and input u variables in a **steady-state**.

If derivatives do not appear in Equation (2.1a), i.e.,

$$g[y(t),u(t)] = 0 \quad \text{or} \quad g(y,u) = 0, \tag{2.7}$$

then it is the mathematical model of the **static system** (it has no memory).

State space mathematical models of a dynamic system are very important. They are used for describing both SISO and MIMO systems.

The state space model of the SISO dynamic system has the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}[\boldsymbol{x}(t), u(t)], \;\; \boldsymbol{x}(0) = \boldsymbol{x}_0 \quad \textbf{– state equation} \tag{2.8a}$$

$$y(t) = h[\boldsymbol{x}(t), u(t)] \quad \textbf{– output equation} \tag{2.8b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, ..., x_n]^T,$$

$$\boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = [g_1, g_2, ..., g_n]^T,$$

where $\boldsymbol{x}(t)$ is the **state vector** (state) of the dimension n, $\boldsymbol{g}$ – the generally nonlinear function of the dimension n, h – the generally nonlinear function, T – the transposition symbol.

We often omit the independent variable time t in order to simplify a description.

The components $x_1, x_2, \ldots, x_n$ of the state $\boldsymbol{x}$ express the inner variables. Knowledge of them is very important for **state space control** (see Chapter 4).

The system order n is given by the number of state variables. If in the output equation the input $u(t)$ does not appear then the given dynamic system (2.8) is strongly physically realizable. In other cases, it is only weakly physically realizable.

The static characteristic (if it exists) from the state space model (2.8) can be obtained for $t \to \infty \;\Rightarrow \dot{\boldsymbol{x}}(t) \to \boldsymbol{0}$ and by the elimination of the state variables (see Example 2.1).

Example 2.1

The nonlinear dynamic system is described by the differential equation of the second order

$$a_2 \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}\, y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 \,\mathrm{sign}[u(t)]\sqrt{|u(t)|}\,, \tag{2.9}$$

with initial conditions $y(0) = y_0$ a $\dot{y}(0) = \dot{y}_0$.

It is necessary to:

a) determine the physically realizability,

b) determine and plot the static characteristic,

c) express the mathematical model (2.9) in the form of the state space model.

Solution:

a) Therefore $n = 2 > m = 0$ [on the right side of the differential equation the derivative of $u(t)$ does not appear], the given dynamic system is strongly physically realizable.

b) In the steady-state for $t \to \infty$ the derivatives in the equation (2.9) are zeros, and therefore in accordance with (2.6) we can write

$$a_0 y = b_0 \operatorname{sign}(u)\sqrt{|u|} \;\Rightarrow$$
$$y = \frac{b_0}{a_0}\operatorname{sign}(u)\sqrt{|u|}.$$

The obtained static characteristic is shown in Fig. 2.1.

c) If we choose the state variables, e.g.

$$x_1 = y,$$
$$x_2 = \dot{x}_1 = \dot{y},$$

then after substitution in the equation (2.9) and modification we get

$$\dot{x}_1 = x_2, \qquad x_1(0) = y_0,$$
$$\dot{x}_2 = -\frac{a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2 + \frac{b_0}{a_2}\operatorname{sign}(u)\sqrt{|u|}, \qquad x_2(0) = \dot{y}_0.$$

The static characteristic can be obtained for the steady-state, i.e. $t \to \infty \Rightarrow$ $\dot{x}_1(t) \to 0$, $\dot{x}_2(t) \to 0$ and after elimination of the state variables

$$\left.\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2 + \frac{b_0}{a_2}\operatorname{sign}(u)\sqrt{|u|} \\ y &= x_1 \end{aligned}\right\} \Rightarrow$$

$$y = \frac{b_0}{a_0}\operatorname{sign}(u)\sqrt{|u|}\,.$$

2.2 Linear dynamic models

Linear mathematical models create a very important group of mathematical models of dynamic systems. These mathematical models must satisfy the condition of the **linearity** which consists of two partial properties: **additivity** and **homogeneity**.

Additivity

$$\left.\begin{aligned} u_1 &\to \text{system} \to y_1 \\ u_2 &\to \text{system} \to y_2 \end{aligned}\right\} \Rightarrow u_1 + u_2 \to \text{system} \to y_1 + y_2\,. \qquad (2.10a)$$

Homogeneity:

$$u \to \text{system} \to y \Rightarrow au \to \text{system} \to ay\,. \qquad (2.10b)$$

These partial properties may be joined

$$\left.\begin{array}{l} u_1 \to \text{system} \to y_1 \\ u_2 \to \text{system} \to y_2 \end{array}\right\} \Rightarrow a_1u_1 + a_2u_2 \to \text{system} \to a_1y_1 + a_2y_2, \qquad (2.11)$$

where a, a_1, a_2 are any constants; $u(t)$, $u_1(t)$ and $u_2(t)$ – the input variables (inputs); $y(t)$, $y_1(t)$ and $y_2(t)$ – the output variables (outputs).

The linearity of a dynamic system has such a property when the weighted sum of output variables corresponds to the weighting sum of input variables.

A very important property of linear dynamic systems is: ***every local property they have is at the same time their global property***.

Example 2.2

The static system is described by the linear algebraic equation

$$y(t) = k_1u(t) + y_0, \qquad (2.12)$$

where k_1 and y_0 are constants.

Is it necessary to verify whether the mathematical model (2.12) is linear.

Solution:

We choose, e.g. $u_1(t) = 2$ and $u_2(t) = 4t$.

After adding in (2.12) we obtain

$$\left.\begin{array}{l} u_1(t) = 2\ldots\ \ y_1(t) = 2k_1 + y_0 \\ u_2(t) = 4t\ldots y_2(t) = 4k_1t + y_0 \end{array}\right\} \Rightarrow y_1(t) + y_2(t) = 2k_1(1+2t) + 2y_0,$$

$$u(t) = u_1(t) + u_2(t) = 2(1+2t)\ldots y = 2k_1(1+2t) + y_0 \neq y_1(t) + y_2(t) =$$
$$= 2k_1(1+2t) + 2y_0.$$

We can see that for $y_0 \neq 0$ the mathematical model (2.12) from the point of view of the linearity definition (2.10) or (2.11) is not linear. The mathematical model (2.12) of a static system will be linear only for $y_0 = 0$, see Fig. 2.2.

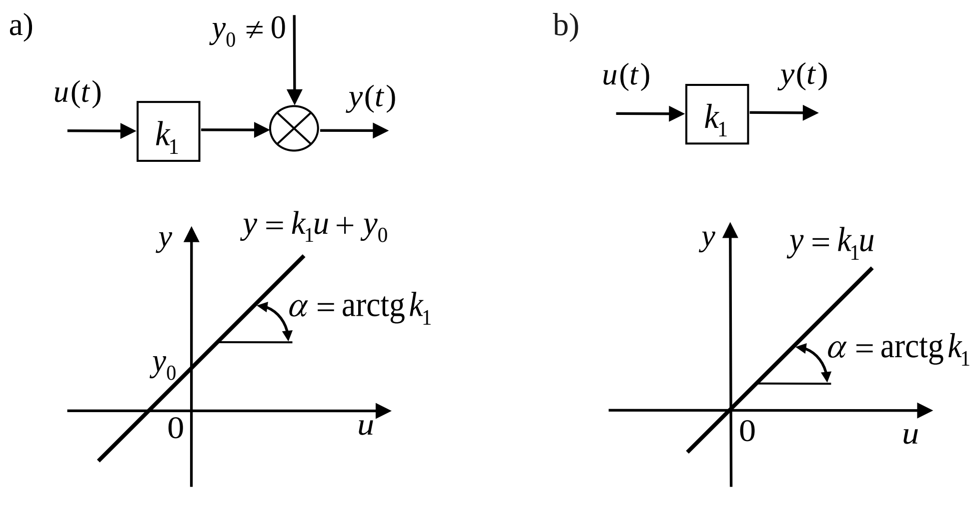


Fig. 2.2 Mathematical model of a static system: a) nonlinear, b) linear – Example 2.2

From the above it is clear that the static characteristic of linear systems (if it exists) must always pass through the origin of coordinates.

Example 2.3

The dynamic system (integrator) is described by the linear differential equation

$$\frac{\mathrm{d}\,y(t)}{\mathrm{d}t} = k_1 u(t),\; y(0) = y_0 \tag{2.13}$$

or the equivalent integral equation

$$y(t) = k_1 \int_0^t u(\tau)\,\mathrm{d}\,\tau + y_0\,. \tag{2.14}$$

It is necessary to verify the linearity of the given mathematical model.

Solution:

We choose the same inputs as in Example 2.2 and we obtain

$$\left.\begin{array}{l} u_1(t) = 2 \ldots\;\; y_1(t) = 2k_1 t + y_0 \\ u_2(t) = 4t \ldots y_2(t) = 2k_1 t^2 + y_0 \end{array}\right\} \Rightarrow y_1(t) + y_2(t) = 2k_1 t(1+t) + 2y_0\,,$$

$$\begin{aligned} &u(t) = u_1(t) + u_2(t) = 2(1+2t) \ldots y = 2k_1 t(1+t) + y_0 \neq y_1(t) + y_2(t) = \\ &= 2k_1 t(1+t) + 2y_0. \end{aligned}$$

We can see again that the mathematical model (2.13) or (2.14) for $y_0 \neq 0$ does not satisfy the condition of the linearity (Fig. 2.3).

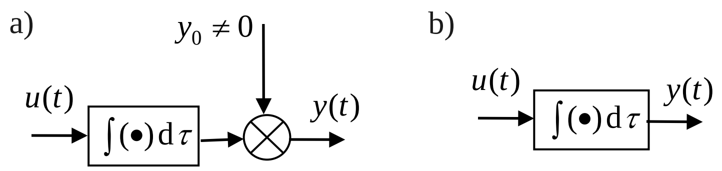


Fig. 2.3 Mathematical model of integrator: a) nonlinear, b) linear – Example 2.3

This particular conclusion can be generalized. ***For linear mathematical models we must always consider zero initial conditions***. Otherwise, we cannot work with them as with mathematical models satisfying the conditions of linearity.

2.3 Basic linear mathematical models

The SISO linear dynamic system in the time domain is very often described by a linear differential equation with constant coefficients (we will consider only such systems)

$$a_n y^{(n)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t) \tag{2.15a}$$

with the initial condition

$$y(0) = y_0, \dot{y}(0) = \dot{y}_0, \ldots, y^{(n-1)}(0) = y_0^{(n-1)} \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0, \ldots, u^{(m-1)}(0) = u_0^{(m-1)} \qquad (2.15b)$$

The conditions of physical realizability are given by the relations (2.2) − (2.4).

When applying the Laplace transform to the differential equation of the *n*-th order (2.15a) with initial conditions (2.15b) we obtain the algebraic equation of the *n*-th degree

$$(a_n s^n + \cdots + a_1 s + a_0)Y(s) - L(s) = (b_m s^m + \cdots + b_1 s + b_0)U(s) - R(s)$$

and from it we can determine the output variable transform

$$Y(s) = \underbrace{\underbrace{\frac{M(s)}{N(s)}U(s)}_{\substack{\text{transform of response}\\ \text{to input}}} + \underbrace{\frac{L(s) - R(s)}{N(s)}}_{\substack{\text{transform of response}\\ \text{to initial conditions}}}}_{\text{transform of solution of differential equation}}, \qquad (2.16)$$

$$M(s) = b_m s^m + \cdots + b_1 s + b_0 = b_m (s - s_1^0)(s - s_2^0)\ldots(s - s_m^0), \qquad (2.17)$$

$$N(s) = a_n s^n + \cdots + a_1 s + a_0 = a_n (s - s_1)(s - s_2)\ldots(s - s_n), \qquad (2.18)$$

where $Y(s)$ is the transform of the output variable $y(t)$, $U(s)$ – the transform of the input variable $u(t)$, $L(s)$ – the polynomial of the max degree $n - 1$ which is determined by the initial conditions of the left side of the differential equation, $R(s)$ – the polynomial of the max degree $m - 1$ which is determined by the initial conditions of the right side of the differential equation, $M(s)$ – the polynomial of the degree m which is determined by the coefficients of the right side of the differential equation, $N(s)$ – the **characteristic polynomial** of the degree n which is determined by the coefficients of the left side of the differential equation, s – the **complex variable** (dimension time^{-1}) [s^{-1}].

Since differential equation (2.15) is the mathematical model of the dynamic system it is obvious that the polynomial $N(s)$ is also at the same time the characteristic polynomial of this dynamic system.

Using the inverse Laplace transform on the transform of the solution (2.16) we obtain the original of the solution

$$y(t) = \mathrm{L}^{-1}\{Y(s)\}. \qquad (2.19)$$

It is very advantageous to use appropriate Laplace transform tables.

From the relation (2.16) it follows that the relation can be used as the linear mathematical model of the given linear dynamic system if the transform of the response at the initial conditions is zero (i.e. the initial conditions are zero), see the conditions of the linearity (2.10) or (2.11). In this case we can write

$$Y(s) = \frac{M(s)}{N(s)}U(s) = G(s)U(s), \qquad (2.20)$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{M(s)}{N(s)}=$$
$$=\frac{b_m s^m+\cdots+b_1 s+b_0}{a_n s^n+\cdots+a_1 s+a_0}=\frac{b_m(s-s_1^0)(s-s_2^0)\ldots(s-s_m^0)}{a_n(s-s_1)(s-s_2)\ldots(s-s_n)}, \tag{2.21}$$

where $G(s)$ is the **transfer function**, s_i – the **poles** of the linear dynamic system = the roots of the characteristic polynomial $N(s)$, s_j^0 – the **zeros** of the linear dynamic system = the roots of the polynomial $M(s)$. The difference $n – m$ is called the **relative degree** of the given system.

The transfer function $G(s)$ ***is given by the ratio of the transform of the output variable*** $Y(s)$ ***and of the transform of the input variable*** $U(s)$ ***for zero initial conditions***. It can be obtained directly from the differential equation (2.15a), because the transforms of the derivatives of the output $y(t)$ and the input $u(t)$ variables for zero initial conditions are given by the simple formulas

$$\left.\begin{aligned} \mathrm{L}\left\{y^{(i)}(t)\right\}&=s^i Y(s); \quad i=1,2,\ldots,n, \\ \mathrm{L}\left\{u^{(j)}(t)\right\}&=s^j U(s); \quad j=1,2,\ldots,m. \end{aligned}\right\} \tag{2.22}$$

The great advantage of the transfer function $G(s)$ is the fact that it allows to express the properties of the linear dynamic system in the complex variable domain by a block as in Fig. 2.4.

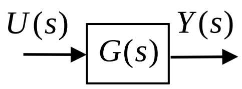


Fig. 2.4 Block diagram of the dynamic system

As it will be shown, it is very simple and effective to work with such blocks.

We can get the static characteristic of the linear dynamic system (if it exists) from the differential equation (2.15a) for

$$\left.\begin{aligned} \lim_{t\to\infty} y^{(i)}(t)&=0; \quad i=1,2,\ldots,n, \\ \lim_{t\to\infty} u^{(j)}(t)&=0; \quad j=1,2,\ldots,m, \end{aligned}\right\} \tag{2.23}$$

e.g.

$$y=k_1 u\,, \tag{2.24a}$$

$$k_1=\frac{b_0}{a_0},\ a_0\neq 0\,, \tag{2.24b}$$

where k_1 is the system (plant) gain.

From comparison (2.21), (2.23) and (2.24) a very important relationship between the time t and the complex variable s follows

$$t\to\infty \ \Leftrightarrow\ s\to 0\,. \tag{2.25}$$

It is clear that on the basis of the relation (2.25) we get the equation of the static characteristic (2.24) from the transfer function (2.21), and therefore it is possible to write

$$y = [\lim_{s \to 0} G(s)]u, \; a_0 \neq 0 \, . \qquad (2.26)$$

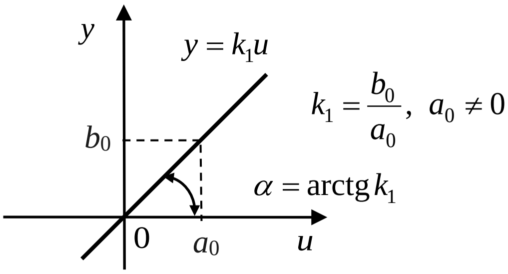


Fig. 2.5 Static characteristic of linear dynamic system

The static characteristic of the linear dynamic system is a straight line which always crosses through the origin of the coordinates (Fig. 2.5).

By substituting complex frequency jω for the complex variable s in the transfer function (2.21) we obtain the **frequency transfer function**

$$G(\mathrm{j}\omega) = G(s)\big|_{s=\mathrm{j}\omega} = \frac{b_m(\mathrm{j}\omega)^m + \cdots + b_1\,\mathrm{j}\omega + b_0}{a_n(\mathrm{j}\omega)^n + \cdots + a_1\,\mathrm{j}\omega + a_0} = A(\omega)\,\mathrm{e}^{\mathrm{j}\varphi(\omega)}\,, \qquad (2.27)$$

$$A(\omega) = \operatorname{mod} G(\mathrm{j}\omega) = \left|G(\mathrm{j}\omega)\right|, \qquad (2.28)$$

$$\varphi(\omega) = \arg G(\mathrm{j}\omega)\,, \qquad (2.29)$$

where $A(\omega)$ is the **modulus** (amplitude, magnitude) of the frequency transfer function, $\varphi(\omega)$ – the **argument** or **phase** of the frequency transfer function, ω – the **angular frequency** (pulsation) (dimension time^{-1}) [s^{-1}].

In order to distinguish angular frequency (T – the period, f – the frequency)

$$\omega = \frac{2\pi}{T}\,, \qquad (2.30)$$

from „ordinary“ frequency

$$f = \frac{1}{T} \qquad (2.31)$$

with the unit [Hz] and the dimension [s^{-1}] for the angular frequency the notation [rad s^{-1}] is used.

The mapping of the frequency transfer function G(jω) for $\omega = 0$ to $\omega = \infty$ in the complex plane is called the **frequency response** (polar plot) (Fig. 2.6).

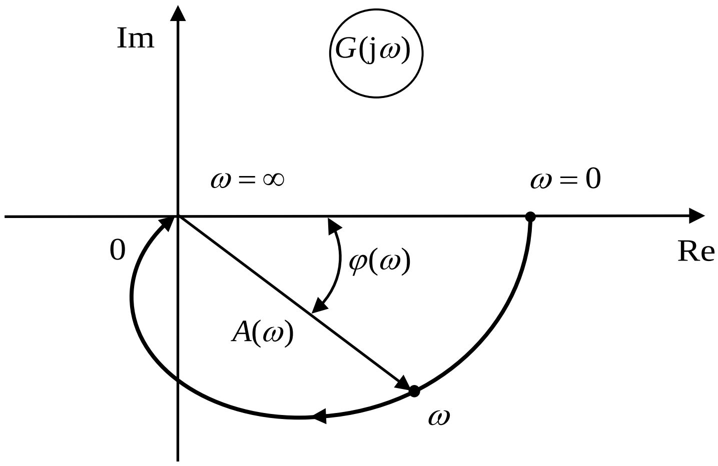


Fig. 2.6 Frequency response

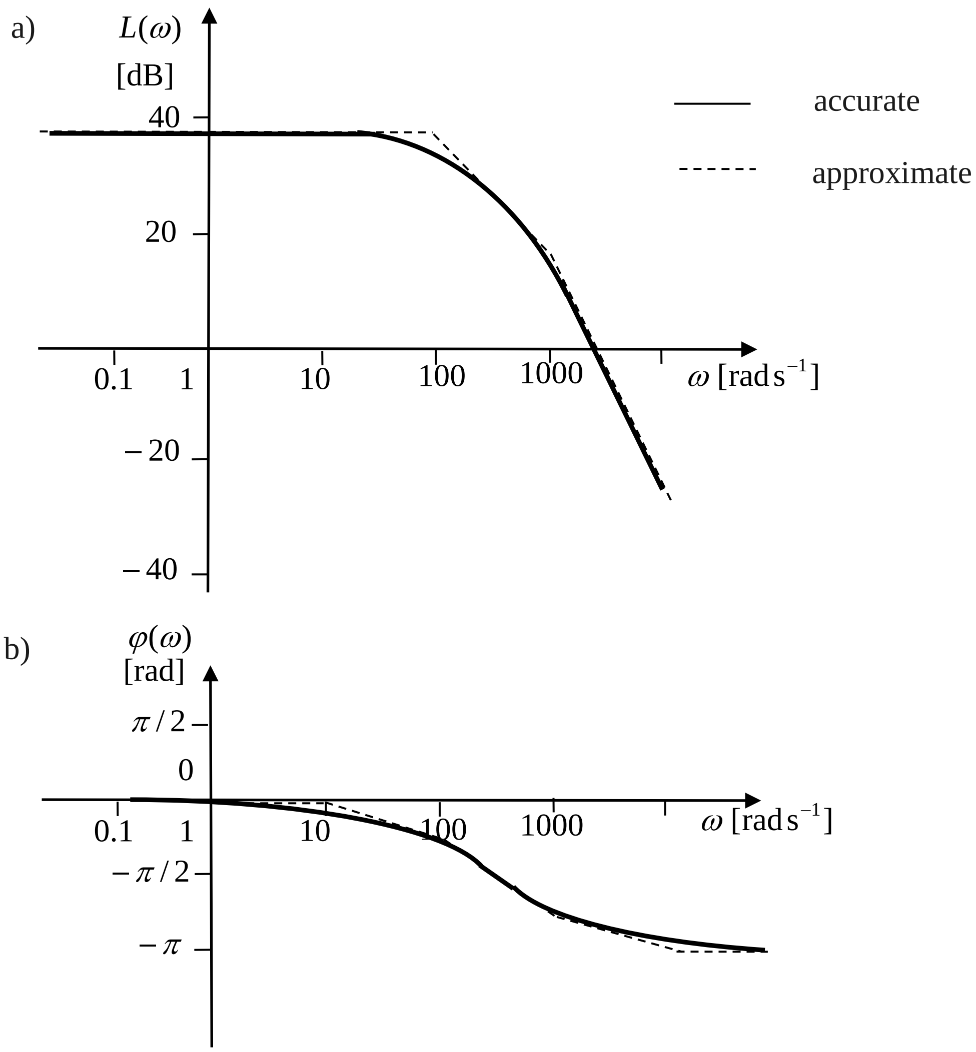


Fig. 2.7 Logarithmic frequency responses: a) Bode magnitude plot, b) Bode phase plot

Logarithmic frequency responses (Bode frequency responses) are most commonly used, see Fig. 2.7. In this case the **Bode magnitude plot**

$$L(\omega) = 20\log A(\omega) \tag{2.32}$$

and the **Bode phase plot** $\varphi(\omega)$ are represented separately. The frequency axis has a logarithmic scale and the logarithmic modulus $L(\omega)$ is given in dB (decibels). For the Bode plots approximations are used on the basis of straight and asymptotic lines.

The frequency transfer function $G(\mathrm{j}\omega)$ expresses for each value of the angular frequency ω the amplitude (modulus, magnitude) $A(\omega)$ and the phase (argument) $\varphi(\omega)$ of the steady-state sinusoidal response $y(t)$ caused by the sinusoidal input $u(t)$ with the unit amplitude.

That means the ***frequency response can be obtained experimentally*** (Fig. 2.8). It has great significance especially for fast systems.

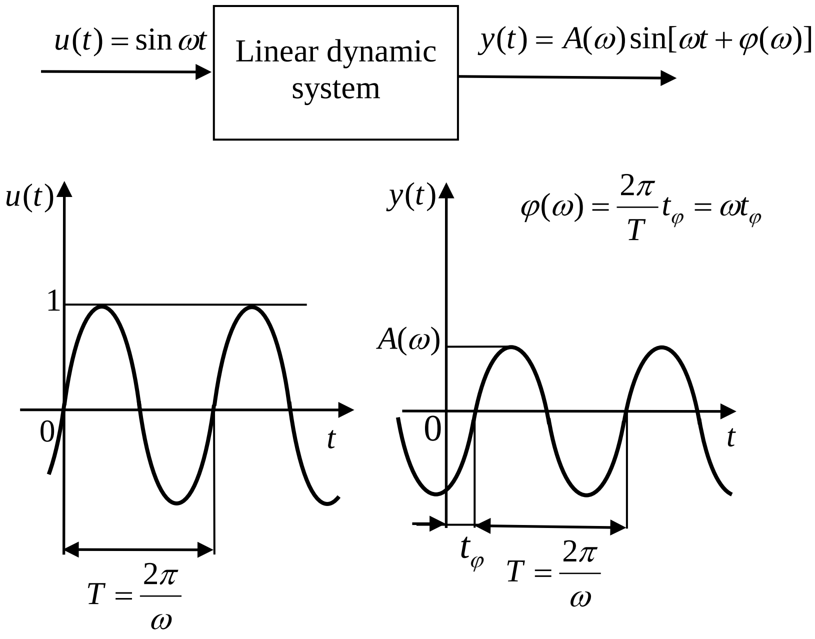


Fig. 2.8 Interpretation of frequency response

The conditions of the physical realizability are given by the relations (2.2) – (2.4). It is obvious that every real dynamic system cannot transfer a signal with an infinitely high angular frequency, therefore for strongly physically realizable dynamic systems there must be held the condition

$$\left.\begin{array}{l} \lim\limits_{\omega\to\infty} G(\mathrm{j}\omega) = 0 \\ \lim\limits_{\omega\to\infty} A(\omega) = 0 \\ \lim\limits_{\omega\to\infty} L(\omega) = -\infty \end{array}\right\} \Leftrightarrow n > m . \tag{2.33}$$

From the frequency transfer function (2.27) we can very easily get the equation of the static characteristic (if it exists) because for the steady-state $\omega = 0$ therefore it must hold

$$y = [\lim_{\omega\to 0} G(\mathrm{j}\omega)]u, \quad a_0 \neq 0 . \tag{2.34}$$

It follows from (2.25) for $s = \mathrm{j}\omega$

$$t \to \infty \;\Leftrightarrow\; \omega \to 0\,. \tag{2.35}$$

It is clear that between the time t and the angular frequency ω the dual relationship holds (Fig. 2.9)

$$t \to 0 \;\Leftrightarrow\; \omega \to \infty\,. \tag{2.36}$$

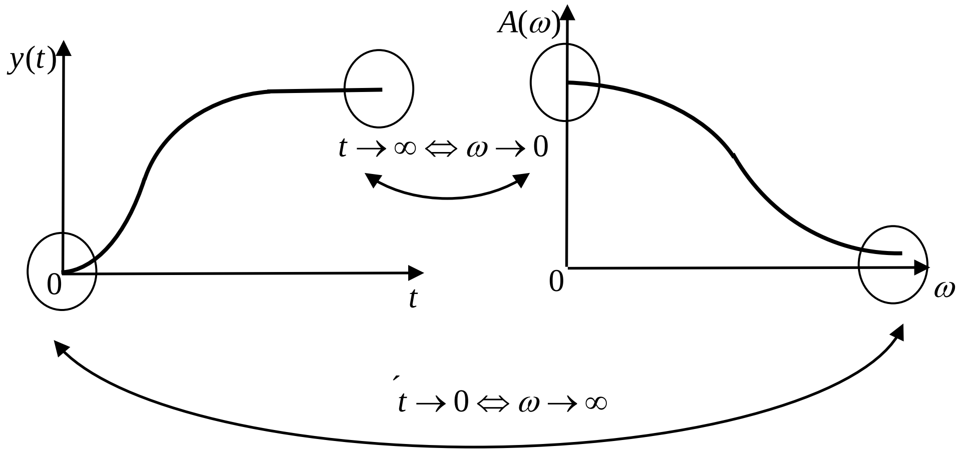


Fig. 2.9 Relationship between the time t and the angular frequency ω

From the relations (2.35), (2.36) and Fig. 2.9 it follows that the properties of the linear dynamic system for low angular frequencies decide about its properties in long periods, i.e. in the steady-states and vice versa. Similarly its properties for high angular frequencies decide about its properties for the initial time response, i.e. about the rise time of the time response (about the transient state) and vice versa.

Properties of linear dynamic systems with zero initial conditions can be expressed by time responses caused by the well-defined courses of an input variable.

In automatic control theory, there are two basic courses of input variable $u(t)$, they are the unit **Dirac impulse** $\delta(t)$ and unit **Heaviside step** $\eta(t)$.

The **impulse response** $g(t)$ describes the response of the linear dynamic system on the input variable in the form of the Dirac impulse $\delta(t)$ for zero initial condition, see Fig. 2.10.

In accordance with the relation (2.20) we can write

$$Y(s) = G(s)U(s) \tag{2.37}$$

and for

$$u(t) = \delta(t) \,\hat{=}\, U(s) = 1$$

we get

$$y(t) = g(t) = \mathrm{L}^{-1}\{G(s)\}. \tag{2.38}$$

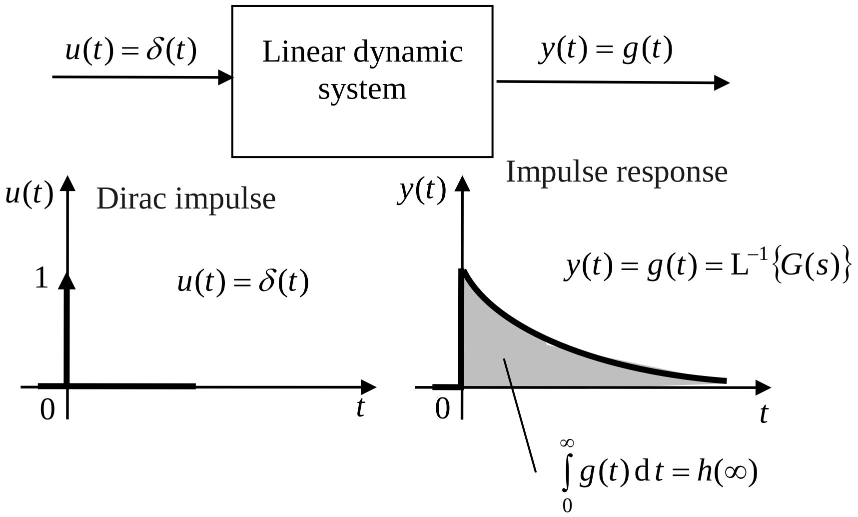


Fig. 2.10 Impulse response of the linear dynamic system

In the linear dynamic system a derivative or an integrating of the input variable *u*(*t*) corresponds to a derivative or an integrating of the output variable *y*(*t*).

We will use these properties for the determination of the static characteristic of the linear dynamic system on the basis of its impulse response *g*(*t*). Since the static characteristic of the linear dynamic system is a straight line crossing through the origin of the coordinates it is enough to determine its one non-zero point. We can write

$$u = u(\infty) = \lim_{t\to\infty}\int_0^t \delta(\tau)\,\mathrm{d}\tau = 1,$$

$$y = y(\infty) = \lim_{t\to\infty}\int_0^t g(\tau)\,\mathrm{d}\tau\,.$$

From this we can easily get the equation of the static characteristic (if it exists)

$$y = [\lim_{t\to\infty}\int_0^t g(\tau)\,\mathrm{d}\tau]u\,. \tag{2.39}$$

The strong condition of the physical realizability has the form

$$|g(0)| < \infty\,. \tag{2.40}$$

If *g*(0) contains the Dirac impulse $\delta(t)$, then the given linear dynamic system is only weakly physically realizable.

The **step response** *h*(*t*) describes the response of the linear dynamic system on the input variable in the form of the Heaviside step $\eta(t)$ for zero initial condition, see Fig. 2.11.

On the basis of the relation (2.37) for

$$u(t) = \eta(t) \mathrel{\hat{=}} U(s) = \frac{1}{s}$$

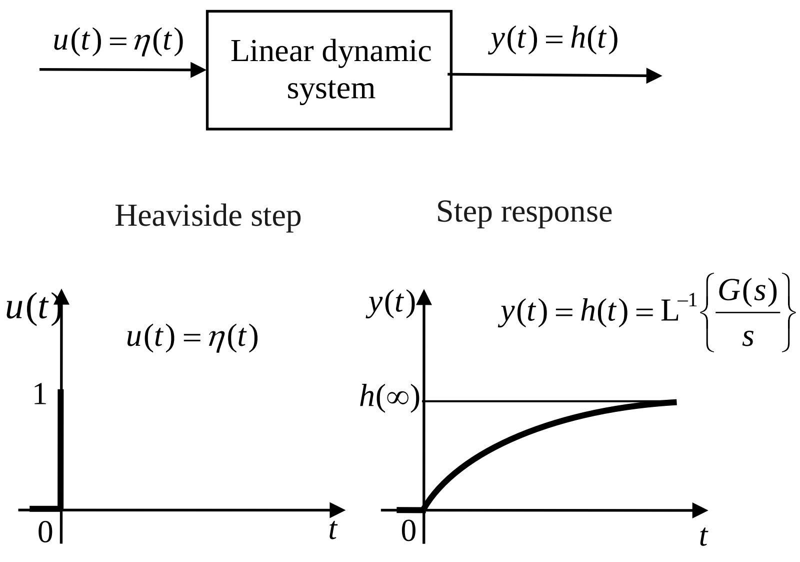


Fig. 2.11 Step response of the linear dynamic system

we get

$$y(t) = h(t) = \mathrm{L}^{-1}\left\{\frac{G(s)}{s}\right\}. \tag{2.41}$$

From the step response $h(t)$ the equation of the static characteristic may be very easily obtained (if it exists) because the relations hold

$$u = u(\infty) = \eta(\infty) = 1,$$

$$y = y(\infty) = h(\infty),$$

i.e.

$$y = [\lim_{t\to\infty} h(t)]u\,. \tag{2.42}$$

The strong condition of the physical realizability has the form

$$h(0) = 0 \tag{2.43}$$

and the weak condition

$$0 < |h(0)| < \infty\,. \tag{2.44}$$

It is useful to apply the **generalized derivative** which is defined by the relations (Fig. 2.12)

$$\left.\begin{aligned} \dot{x}(t) &= \dot{x}_{ob}(t) + \sum_{i=1}^{p} h_i \delta(t - t_i), \\ h_i &= \lim_{t\to t_{i+}} x(t) - \lim_{t\to t_{i-}} x(t), \end{aligned}\right\} \tag{2.45}$$

where t_i are the points of discontinuity with the jumps h_i, $\dot{x}_{or}(t)$ − the ordinary derivative determined between the points of discontinuity.

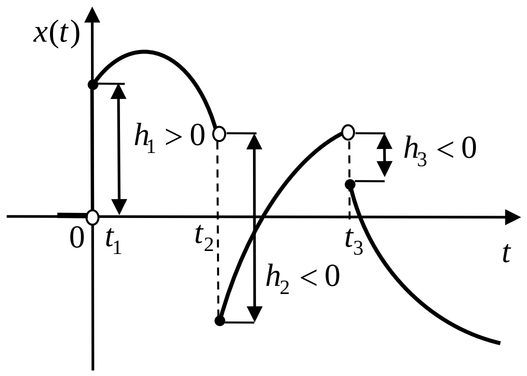


Fig. 2.12 Function $x(t)$ with points of discontinuity

By means of the generalized derivative it is possible to express the relationship between the Dirac impulse and the Heaviside step

$$\delta(t) = \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} \iff \eta(t) = \int_0^t \delta(\tau)d\tau \tag{2.46}$$

and between the impulse and step responses

$$g(t) = \frac{\mathrm{d}h(t)}{\mathrm{d}t} \iff h(t) = \int_0^t g(\tau)d\tau\ , \tag{2.47}$$

$$G(s) = sH(s) \iff H(s) = \frac{G(s)}{s}. \tag{2.48}$$

From all mathematical models of the linear dynamic systems the state space model is the most general

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \quad - \text{state equation} \tag{2.49a}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) + du(t) \quad - \text{output equation} \tag{2.49b}$$

where $\boldsymbol{A}$ is the square system (dynamics) matrix of the order n [$(n \times n)$], $\boldsymbol{b}$ – the vector of the input of the dimension n, $\boldsymbol{c}$ – the vector of the output of the dimension n, d – the transfer constant, T – the transposition symbol.

The block diagram of the state space model of the linear dynamic system (3.35) is in Fig. 2.13.

For $d = 0$ the state space model (2.49) satisfies the strong condition of the physical realizability and for $d \neq 0$ satisfies only the weak condition of physical realizability.

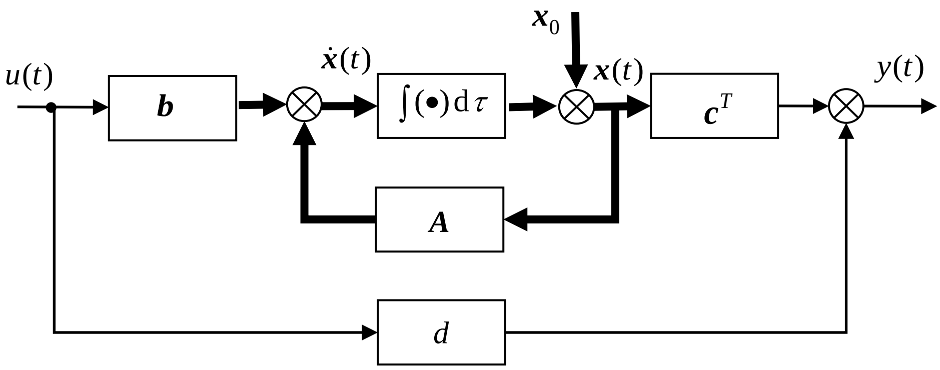


Fig. 2.13 Block diagram of the state space model of the linear dynamic system

If the state space model (2.49) satisfies the **controllability condition** (see Appendix C)

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b},\dots,\boldsymbol{A}^{n-1}\boldsymbol{b}], \quad \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) \neq 0 \tag{2.50}$$

and the **observability condition** (see Appendix C)

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T\boldsymbol{A}^{n-1} \end{bmatrix} = [\boldsymbol{c}, \boldsymbol{A}^T\boldsymbol{c},\dots,(\boldsymbol{A}^T)^{n-1}\boldsymbol{c}]^T, \quad \det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) \neq 0\,, \tag{2.51}$$

then for zero initial conditions [$\boldsymbol{x}(0) = \boldsymbol{x}_0 = \boldsymbol{0}$] we can get the transfer function on the basis of the Laplace transform

$$\left.\begin{aligned} s\boldsymbol{X}(s) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{b}U(s) \\ Y(s) &= \boldsymbol{c}^T\boldsymbol{X}(s) + dU(s) \end{aligned}\right\} \Rightarrow$$

$$G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d\,, \tag{2.52}$$

where det is the determinant, $\boldsymbol{I}$ – the unit matrix, $\boldsymbol{Q}_{co}$ – the **controllability matrix** of order n [$(n\times n)$], $\boldsymbol{Q}_{ob}$ – the **observability matrix** of order n [$(n\times n)$].

From the transfer function (2.52) on the basis of (2.26) we can obtain the equation of the static characteristic (if it exists)

$$y = \lim_{s\to 0}[\boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d]u\,. \tag{2.53}$$

It is preferable for getting the transfer function to use the relation

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) - \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} + d\,, \tag{2.54}$$

which does not demand the inversion of the functional matrix.

Transfer function (2.52) or (2.54) are determined on the basis of the state space model (2.49) uniquely. In contrast to the transfer function

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b'_m s^m+\ldots+b'_1 s+b'_0}{a'_n s^n+\ldots+a'_1 s+a'_0} \tag{2.55a}$$

the state space model can have many (theoretically infinitely many) different forms. For example, for $n = m$ the transfer function (2.55a) can be written down in the form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b'_n}{a'_n}+\frac{b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=$$

$$=d+\frac{b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{N(s)}, \tag{2.55b}$$

$$N(s)=\det(s\boldsymbol{I}-\boldsymbol{A})=s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0 . \tag{2.55c}$$

It is important that the transfer function (2.55) for $d = 0$ has not been possible to simplify by the compensation (cancellation), i.e. the transfer function must be irreducible. In this case we say that the mathematical model has a **minimal form**. Minimal form also state models have derived therefrom. It is obvious that ***controllable and observable linear dynamic systems have a minimal form***.

From the above mentioned mathematical models the state space model is the most general. Assuming controllability and observability [see relations (2.50) and (2.51)] and, of course, zero initial conditions, all these mathematical models of the linear dynamical systems, i.e., linear differential equations, transfer functions, frequency transfer functions, impulse responses, step responses and linear state space models are equivalent and mutually transferable.

Example 2.4

The linear dynamic system is described by the state model

$$\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=-2x_2+u,\\ y&=2x_1.\end{aligned} \tag{2.56}$$

Assuming zero initial conditions, it is necessary to determine: a) the transfer function, b) the frequency transfer function, c) the impulse response, d) the step response.

Solution:

First, it is necessary to verify the controllability and observability of the given system. In accordance with (2.49) and (2.56) we can write

$$\boldsymbol{A}=\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \boldsymbol{c}^T=[2,0],\quad d=0 .$$

Controllability (2.50)

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})=[\boldsymbol{b},\boldsymbol{A}\boldsymbol{b}]=\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix},\quad \det\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})=1\neq 0\ \Rightarrow$$

The linear dynamic system (2.56) is controllable.

Observability (2.51)

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=\begin{bmatrix}\boldsymbol{c}^T\\ \boldsymbol{c}^T\boldsymbol{A}\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\quad \det\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=4\neq 0\ \Rightarrow$$

The linear dynamic system (2.56) is observable.

a) Transfer function

On the basis of the relation (2.52) we can write

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{1}{s(s+2)}\begin{bmatrix}s+2 & 1\\ 0 & s\end{bmatrix},$$

$$G(s)=\boldsymbol{c}^T(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}=\frac{1}{s(s+2)}[2,0]\begin{bmatrix}s+2 & 1\\ 0 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\frac{2}{s(s+2)}.$$

If we use the relation (2.54) we do not need to invert the matrix, i.e. we can write

$$\det(s\boldsymbol{I}-\boldsymbol{A})=\det\begin{bmatrix}s & -1\\ 0 & s+2\end{bmatrix}=s(s+2),$$

$$\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{c}^T)=\det\left\{\begin{bmatrix}s & -1\\ 0 & s+2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}[2,0]\right\}=\det\begin{bmatrix}s & -1\\ 2 & s+2\end{bmatrix}=s(s+2)+2,$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{c}^T)-\det(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{2}{s(s+2)}.$$

We see that we have obtained identical results and that the transfer function has a minimal form.

b) Frequency transfer function

In accordance with the relation (2.27) we can write directly

$$G(\mathrm{j}\omega)=G(s)\big|_{s=\mathrm{j}\omega}=\frac{2}{\mathrm{j}\omega(\mathrm{j}\omega+2)}=-\frac{2}{\omega^2+4}-\mathrm{j}\frac{4}{\omega(\omega^2+4)}.$$

The frequency response is shown in Fig. 2.14a.

c) Impulse response

On the basis of the relation (2.38) we get

$$g(t)=\mathrm{L}^{-1}\{G(s)\}=\mathrm{L}^{-1}\left\{\frac{2}{s(s+2)}\right\}=1-\mathrm{e}^{-2t}.$$

The impulse response is shown in Fig. 2.14b.

d) Step response

On the basis of the relation (2.41) we get

$$h(t)=\mathrm{L}^{-1}\left\{\frac{G(s)}{s}\right\}=\mathrm{L}^{-1}\left\{\frac{2}{s^2(s+2)}\right\}=t+\frac{1}{2}\left(\mathrm{e}^{-2t}-1\right).$$

The step response is shown in Fig. 2.14c.

We will verify yet a connection between the impulse and the step responses on the basis of the relations (2.47) and (2.48), i.e.

$$g(t) = \frac{\mathrm{d}\, h(t)}{\mathrm{d}\, t} = \frac{\mathrm{d}}{\mathrm{d}\, t}\left[t + \frac{1}{2}\left(\mathrm{e}^{-2t} - 1\right)\right] = 1 - \mathrm{e}^{-2t},$$

$$h(t) = \int_0^t g(\tau)\,\mathrm{d}\,\tau = \int_0^t \left(1 - \mathrm{e}^{-2\tau}\right)\mathrm{d}\,\tau = t + \frac{1}{2}[\mathrm{e}^{-2\tau}]_0^t = t + \frac{1}{2}\left(\mathrm{e}^{-2t} - 1\right).$$

We see that the relations (2.47) and (2.48) really hold.

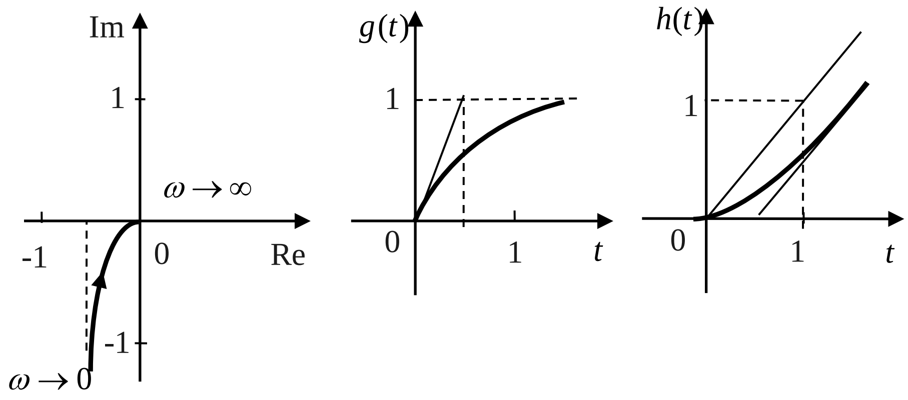


Fig. 2.14 Responses: a) frequency, b) impulse, c) step – Example 2.4

Example 2.5

The transfer function of a conventional PI controller is given by

$$G_C(s) = \frac{U(s)}{E(s)} = K_P + K_I\,\frac{1}{s}, \tag{2.57}$$

where $U(s)$ is the transform of the manipulated variable, $E(s)$ – the transform of the control error, K_P – the weight of the proportional component, K_I – the weight of the integral component. The transfer function of the PI controller should be expressed in the form of a state model.

Solution:

The transfer function of the PI controller we express in the time domain in the form of the integrodifferential equation

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\,\mathrm{d}\,\tau\ .$$

If we choose as the state variable

$$x(t) = \int_0^t e(\tau)\,\mathrm{d}\,\tau\ ,$$

then we can write

$$\begin{aligned} \dot{x}(t) &= e(t), \\ u(t) &= K_I x(t) + K_P e(t). \end{aligned} \tag{2.58}$$

We obtained the simple state model of the PI controller, see Fig. 2.15.

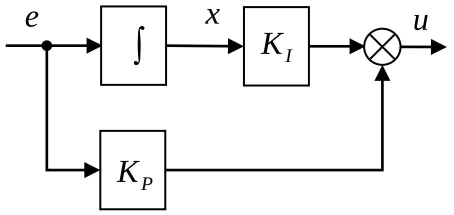


Fig. 2.15 State model of PI controller – Example 2.5

3 STATE MODELS OF LINEAR DYNAMIC SYSTEMS

3.1 Asymptotic stability

The stability of the linear dynamic systems is their most important property. It must be understood as the ability of the dynamic systems to stabilize all variables on the finite values, if all input variables are fixed at finite values.

Consider a linear dynamic system described by the state model [see (2.49)]

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.1a}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) + du(t). \tag{3.1b}$$

Because the output equation (3.1b) is algebraic (static) the stability is determined by the state (dynamic) equation (3.1a).

The necessary and sufficient condition for the asymptotic stability of the linear dynamic system (3.1) is that the roots $s_1, s_2, \ldots, s_n$ of its characteristic polynomial [see (2.55)]

$$\begin{aligned} N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) &= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = \\ &= (s - s_1)(s - s_2)\ldots(s - s_n) \end{aligned} \tag{3.2}$$

have a negative real part, i.e.

$$\operatorname{Re} s_i < 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{3.3}$$

It is clear that the roots $s_1, s_2, \ldots, s_n$ are simultaneously the poles of the given system (3.1) [see (2.55)] and also **eigenvalues** of the matrix $\boldsymbol{A}$.

For the asymptotically stable linear dynamic system a static characteristic must exist.

To verify the asymptotic stability of the linear dynamic system with the state model (3.1) any criterion based on the characteristic polynomial (3.2) can be applied.

Example 3.1

It is necessary to verify the asymptotic stability of the linear dynamic system (2.56) from the Example 2.4.

Solution:

In the Example 2.4 has been determined the characteristic polynomial

$$N(s) = s(s+2) \ \Rightarrow \ s_1 = 0, \ s_1 = -2.$$

Because one pole is zero it is clear that the linear dynamic system is not asymptotically stable. From the viewpoint of the linear theory, the given dynamic system is on the stability boundary and from the viewpoint of the Lyapunov theory it is stable.

Example 3.2

The mathematical model of the DC motor with a constant separate excitation (furthermore, we will use “DC motor”) is shown in Fig. 3.1, where means: J_m – the total moment of inertia reduced in the motor shaft [kg m^2], $i_a(t)$ – the armature current [A], $u_a(t)$ – the armature voltage [V], R_a – the total resistance of the armature circuit [Ω], L_a – the total inductance of the armature circuit [H], b_m – the coefficient of viscous friction [N m s rad^{-1}], $m(t)$ – the motor torque [N m], $m_l(t)$ – the load torque [N m], $\alpha(t)$ – the angle of the motor shaft [rad], $\omega(t)$ – the angular velocity of the motor shaft [rad s^{-1}], c_m – the motor constant [N m A^{-1}], c_e – the motor constant [V s rad^{-1}], $u_e(t)$ – the induced voltage [V], Φ – the constant magnetic flux of the excitation [Wb].

It is necessary to derive a DC motor state model assuming that the output variables are the angle $\alpha(t)$ and angular velocity $\omega(t)$. In the state model with output $\omega(t)$ it is required to verify the asymptotic stability.

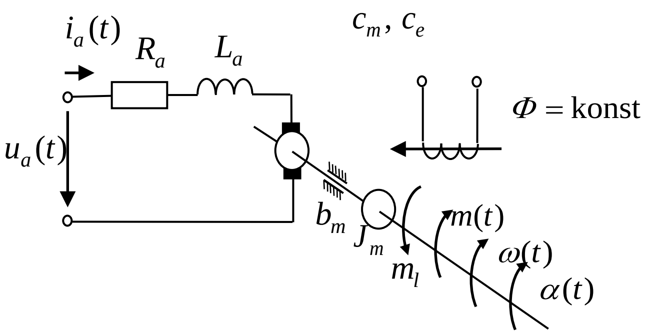


Fig. 3.1 Simplified scheme of the DC motor – Example 3.2

Solution:

In accordance with Fig. 3.1, we can write:

$$\left.\begin{aligned}
\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} &= \omega(t),\\
J_m\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} + b_m\omega(t) &= m(t) - m_l(t),\\
m(t) &= c_m i_a(t),\\
L_a\frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) &= u_a(t) - u_e(t),\\
u_e(t) &= c_e\omega(t).
\end{aligned}\right\} \tag{3.4}$$

We can get the state model of the DC motor with separate excitation from the equations (3.4), i.e.

$$\left.\begin{aligned}
\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} &= \omega(t),\\
\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} &= -\frac{b_m}{J_m}\omega(t)+\frac{c_m}{J_m}i_a(t)-\frac{1}{J_m}m_l(t),\\
\frac{\mathrm{d}i_a(t)}{\mathrm{d}t} &= -\frac{c_e}{L_a}\omega(t)-\frac{R_a}{L_a}i_a(t)+\frac{1}{L_a}u_a(t).
\end{aligned}\right\} \tag{3.5}$$

The system of the equations (3.5) we write in the matrix form

$$\begin{bmatrix}\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}\\ \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\\ \frac{\mathrm{d}i_a(t)}{\mathrm{d}t}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{b_m}{J_m} & \frac{c_m}{J_m}\\ 0 & -\frac{c_e}{L_a} & -\frac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}\alpha(t)\\ \omega(t)\\ i_a(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{L_a}\end{bmatrix}u_a(t)-\begin{bmatrix}0\\ \frac{1}{J_m}\\ 0\end{bmatrix}m_l(t)\,. \tag{3.6}$$

The state model (3.6) [or (3.5)] applies to the output $\alpha(t)$. Without the first equation in (3.5) we get the state model for the output $\omega(t)$

$$\begin{bmatrix}\frac{\mathrm{d}\,\omega(t)}{\mathrm{d}t}\\ \frac{\mathrm{d}\,i_a(t)}{\mathrm{d}t}\end{bmatrix} = \begin{bmatrix}-\frac{b_m}{J_m} & \frac{c_m}{J_m}\\ -\frac{c_e}{L_a} & -\frac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}\omega(t)\\ i_a(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{L_a}\end{bmatrix}u_a(t)-\begin{bmatrix}\frac{1}{J_m}\\ 0\end{bmatrix}m_l(t)\,. \tag{3.7}$$

For powers at a steady state the equality holds, i.e.

$$u_e i_a = m\omega \;\Rightarrow\; c_e\omega i_a = c_m i_a\omega \;\Rightarrow\; c_e = c_m.$$

It is necessary to verify the asymptotic stability of the DC motor with the state model (3.7), and therefore we can write ($c_e = c_m$)

$$\boldsymbol{A} = \begin{bmatrix}-\frac{b_m}{J_m} & \frac{c_m}{J_m}\\ -\frac{c_m}{L_a} & -\frac{R_a}{L_a}\end{bmatrix},$$

$$N_\omega(s) = \det(s\boldsymbol{I}-\boldsymbol{A}) = \begin{bmatrix}s+\frac{b_m}{J_m} & -\frac{c_m}{J_m}\\ \frac{c_m}{L_a} & s+\frac{R_a}{L_a}\end{bmatrix} = \left(s+\frac{b_m}{J_m}\right)\left(s+\frac{R_a}{L_a}\right)+\frac{c_m^2}{J_m L_a} =$$

$$= s^2+\left(\frac{b_m}{J_m}+\frac{R_a}{L_a}\right)s+\frac{c_m^2+R_a b_m}{J_m L_a} \;\Rightarrow\; \operatorname{Re}s_1<0,\ \operatorname{Re}s_2<0. \tag{3.8}$$

Because the characteristic polynomial of the second degree has the positive coefficients, therefore based on necessary and sufficient Stodola‘s stability conditions, the linear dynamic system representing the DC motor, for the output shaft angular velocity $\omega(t)$ is asymptotically stable.

It is easy to show that for the output angular shaft velocity $\alpha(t)$ the linear dynamic system (3.6) will have the characteristic polynomial

$$N_\alpha(s) = s\left[s^2 + \left(\frac{b_m}{J_m} + \frac{R_a}{L_a}\right)s + \frac{c_m^2 + R_a b_m}{J_m L_a}\right] = sN_\omega(s) \Rightarrow$$
$$\Rightarrow s_1 = 0, \operatorname{Re} s_2 < 0, \operatorname{Re} s_3 < 0. \tag{3.9}$$

In this case, the DC motor with a constant separate excitation is not asymptotically stable. Similarly as in Example 3.1, from the viewpoint of the linear theory the given linear dynamic system is on the stability boundary and from the viewpoint of the Lyapunov theory it is stable.

3.2 Controllability and observability

Mathematical models in the form of the transfer function, the frequency transfer function, the impulse response and the step response describe uniquely the behaviour of the controllable and observable linear dynamic system with zero initial conditions (for more details see Appendix C).

For the state model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t),$$
$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) + du(t) \tag{3.10}$$

the controllability condition (2.50) $\det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \neq 0$ expresses a very important property of the linear dynamic system consisting in the fact that there exists such an input (control) variable $u(t)$ which transfers the system from any initial state $\boldsymbol{x}(t_0)$ to any given final state $\boldsymbol{x}(t_1)$ in finite time $t_1 - t_0$. Most often, it is assumed that the final state is the origin, i.e. $\boldsymbol{x}(t_1) = \boldsymbol{0}$.

On the other hand the observability condition (2.51) $\det \boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) \neq 0$ indicates that on the basic of the input (control) $u(t)$ and output $y(t)$ variables courses given on the finite time interval $t_1 - t_0$ the initial state $\boldsymbol{x}(t_0)$ can be determined.

The linear dynamic system with the state model (3.10) can be divided into four parts (it is so called the **Kalman decomposition of system**) in accordance with Fig. 3.2:

controllable and observable part,

controllable and unobservable part,

uncontrollable and observable part,

uncontrollable and unobservable part.

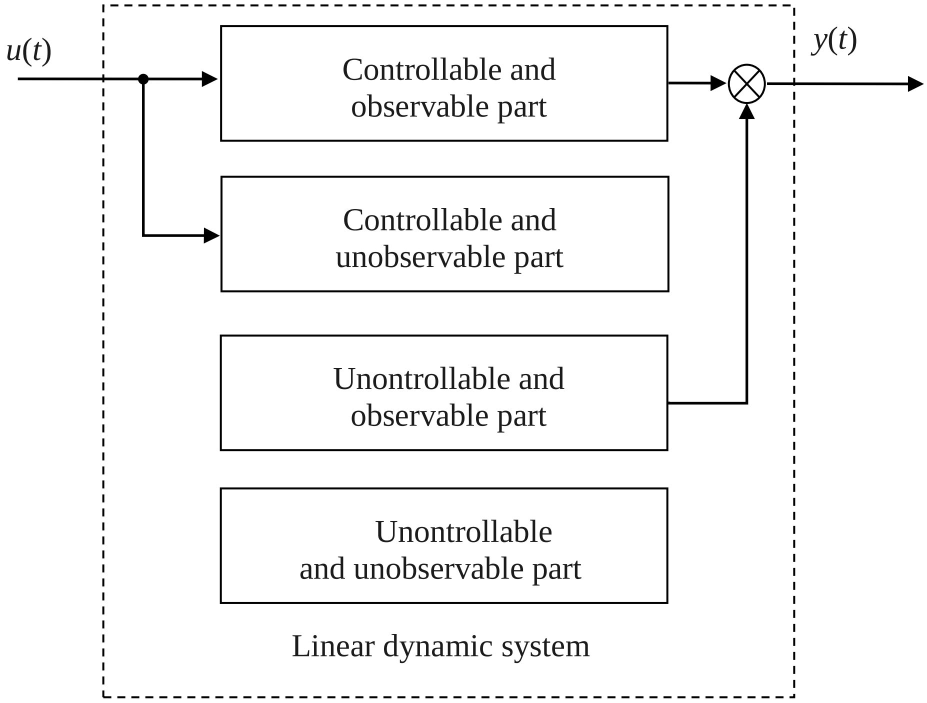


Fig. 3.2 Kalman decomposition of linear dynamic system

For a technical practice, it is very important that uncontrollable and unobservable parts are asymptotically stable. If the uncontrollable part is asymptotically stable, then the linear dynamical system is **stabilizable** and if an unobservable part is asymptotically stable, then the linear dynamical system is **detectable**.

Example 3.3

For the linear dynamic system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t), \\ \dot{x}_2(t) &= -2x_2(t) + u(t), \\ \dot{x}_3(t) &= 0, \\ y(t) &= x_1(t) + x_3(t) \end{aligned} \tag{3.11}$$

it is necessary to carry out the Kalman decomposition.

Solution:

In accordance with (3.11) we can write

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}^T = [1,0,1], \quad d = 0 .$$

Controllability (2.50)

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{Ab}, \boldsymbol{A}^2\boldsymbol{b}] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = 0 .$$

The linear dynamic system (3.11) is uncontrollable.

Observability (2.51)

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=\begin{bmatrix}\boldsymbol{c}^T\\ \boldsymbol{c}^T\boldsymbol{A}\\ \boldsymbol{c}^T\boldsymbol{A}^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ -1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix},\ \ \det\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=0\,.$$

The linear dynamic system (3.10) is unobservableable.

On the basis of the state model (3.11) we can build the block diagram in Fig. 3.3 which indicates that the state variable $x_2(t)$ is unobservable and state variable $x_3(t)$ is uncontrollable. From Fig. 3.3 it is obvious that the poles of the system are $s_1=-1$, $s_2=-2$ and $s_3=0$, i.e. the linear dynamic system is uncontrollable and unstabilizable, unobservable but detectable (the observable part is asymptotically stable, while the uncontrollable part is not asymptotically stable).

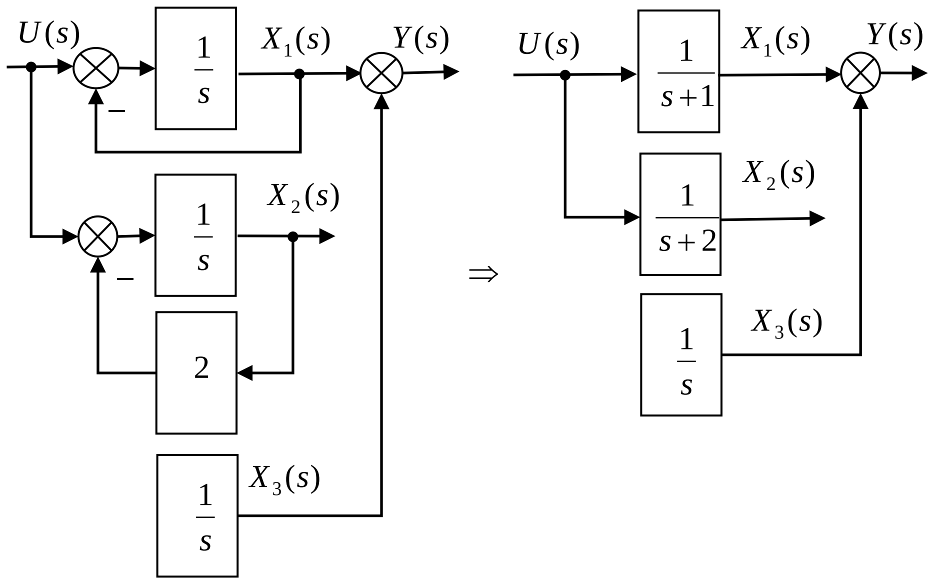


Fig. 3.3 Kalman decomposition – Example 3.3

We determine the transfer function of the state model (3.11) on the basis of the relation (2.54)

$$\det(s\boldsymbol{I}-\boldsymbol{A})=s(s+1)(s+2),$$

$$\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{c}^T)=s(s+2)^2,$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{c}^T)-\det(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=$$

$$=\frac{s(s+2)}{s(s+1)(s+2)}=\frac{1}{(s+1)}.$$

It is obvious that the state model (3.11) had not a minimal form because in the transfer function the compensation (cancellation) occured, and thus the order of the system reduced from 3 to 1.

Example 3.4

It is necessary to verify the controllability and observability of the nonlinear dynamic system which is described by the state model

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t), \\ \dot{x}_2(t) &= -x_2(t) + u(t), \\ y(t) &= x_1(t) + x_2(t). \end{aligned} \qquad (3.12)$$

Solution:

From the state model (3.12) we get

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{c}^T = [1,1], \quad d = 0 .$$

Controllability (2.50)

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = 0 .$$

The linear dynamic system (3.12) is uncontrollable.

Observability (2.51)

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = 0 .$$

The linear dynamic system (3.12) is unobservableable.

On the basis of the equations (3.12) the block diagram of the linear dynamic system can be built, Fig. 3.4.

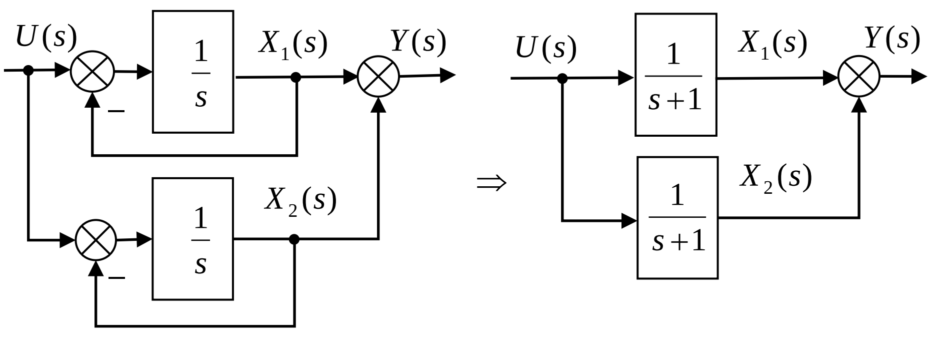


Fig. 3.4 Block diagram of linear dynamic system – Example 3.4

From Fig. 3.4 it follows that both state variables $x_1(t)$ and $x_2(t)$ are equal, and therefore any state $\boldsymbol{x}(t) = [x_1(t), x_2(t)]^T$ in the state plane (x_1, x_2) cannot be achieved by the input (control) $u(t)$. It is also obvious that these state variables cannot be distinguished from each other and therefore they are also unobservable. Because the poles are $s_1 = s_2 = -1$, the linear dynamic system is asymptotically stable, and therefore, even when it is uncontrollable and unobservable, it is stabilizable and detectable, and therefore it is practically usable.

The given linear dynamic system is of the second order, but from the outside view it seems as the system of the first order with the transfer function

$$\frac{Y(s)}{U(s)} = \frac{2}{s+1}.$$

3.3 Basic canonical forms

Consider a linear dynamic system whose state model has the general form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \\ y(t) &= \boldsymbol{c}^T \boldsymbol{x}(t) + du(t), \end{aligned} \tag{3.13a}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}, \ \boldsymbol{c}^T = [c_{11}, c_{12}, \dots, c_{1n}]. \tag{3.13b}$$

The vectors $\boldsymbol{b}$ and $\boldsymbol{c}^T$ have two indeces because the vector $\boldsymbol{b}$ is the first column in the general input matrix $\boldsymbol{B}$ and the vector $\boldsymbol{c}^T$ is the first row in the general output matrix $\boldsymbol{C}$ for MIMO linear dynamical systems.

In the text for clarity a dependence on the time t is not explicitly expressed, as well we will talk simply about a system (the terms a model and system will be considered as equivalent) and indices will be used: t – transformation, co – controllability, c – control, controller, ob – observability, o – observe, observer, d – diagonal.

Further it is assumed that the linear dynamic system (3.13) is controllable and observable, i.e. the conditions (2.50) and (2.51) hold (it has a minimal form)

$$\det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \neq 0 \quad \text{a} \quad \det \boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) \neq 0.$$

If we introduce the regular square transformation matrix $\boldsymbol{T}_t$ of the order n by the relation

$$\boldsymbol{x} = \boldsymbol{T}_t \boldsymbol{x}_t, \quad \det \boldsymbol{T}_t \neq 0, \tag{3.14}$$

then the state model (3.13) can be transformed from the state space X into the new state space X_t, i.e. we obtain the transformed state model

$$\begin{aligned} \dot{\boldsymbol{x}}_t &= \boldsymbol{A}_t \boldsymbol{x}_t + \boldsymbol{b}_t u, \\ y &= \boldsymbol{c}_t^T \boldsymbol{x}_t + du, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{T}_t^{-1} \boldsymbol{x}, \\ \boldsymbol{A}_t &= \boldsymbol{T}_t^{-1} \boldsymbol{A} \boldsymbol{T}_t, \\ \boldsymbol{b}_t &= \boldsymbol{T}_t^{-1} \boldsymbol{b}, \\ \boldsymbol{c}_t^T &= \boldsymbol{c}^T \boldsymbol{T}_t. \end{aligned} \tag{3.16}$$

The transfer constant d remains unchanged after the transformation.

Both system (dynamics) matrices $\boldsymbol{A}$ and $\boldsymbol{A}_t$ are similar because they have the same characteristic polynomials, and hence the same eigenvalues, i.e.

$$\begin{aligned} N(s) &= \det(s\boldsymbol{I}-\boldsymbol{A}_t) = \det(s\boldsymbol{I}-\boldsymbol{T}_t^{-1}\boldsymbol{A}\boldsymbol{T}_t) = \\ &= \det[\boldsymbol{T}_t^{-1}(s\boldsymbol{I}-\boldsymbol{A})\boldsymbol{T}_t] = \\ &= \det\boldsymbol{T}_t^{-1}\det(s\boldsymbol{I}-\boldsymbol{A})\det\boldsymbol{T}_t = \det(s\boldsymbol{I}-\boldsymbol{A}) = \\ &= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 . \end{aligned} \tag{3.17}$$

Thus this transformation is called the **similarity transformation**.

Canonical controller form

For the transformation matrix

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c,\boldsymbol{b}_c), \tag{3.18a}$$

where

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c,\boldsymbol{b}_c) = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{3.18b}$$

on the basis of the relations (3.15) and (3.16) we get (index t is necessary to replace by the index c) the **canonical** (normal) **controller form**

$$\begin{aligned} \dot{\boldsymbol{x}}_c &= \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{b}_c u, \\ y &= \boldsymbol{c}_c^T\boldsymbol{x}_c + du , \end{aligned} \tag{3.19a}$$

where

$$\begin{aligned} \boldsymbol{A}_c &= \boldsymbol{T}_c^{-1}\boldsymbol{A}\boldsymbol{T}_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \\ \boldsymbol{b}_c &= \boldsymbol{T}_c^{-1}\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{c}_c^T = \boldsymbol{c}^T\boldsymbol{T}_c = [b_0, b_1, \ldots, b_{n-1}] . \end{aligned} \tag{3.19b}$$

The square matrix (3.18b) is the inverse of the controllability matrix of the canonical controller form (3.19)

$$\boldsymbol{Q}_{co}(\boldsymbol{A}_c,\boldsymbol{b}_c) = [\boldsymbol{b}_c, \boldsymbol{A}_c\boldsymbol{b}_c, \ldots, \boldsymbol{A}_c^{n-1}\boldsymbol{b}_c], \tag{3.20}$$

for which it holds

$$\left|\det \boldsymbol{Q}_{co}(\boldsymbol{A}_c, \boldsymbol{b}_c)\right| = \left|\det \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c)\right| = 1 . \tag{3.21}$$

It can be easily proved. Let's multiple both sides of the equation (3.18a) from the right by the matrix $\boldsymbol{Q}_{co}(\boldsymbol{A}_c, \boldsymbol{b}_c)$, i.e.

$$\boldsymbol{T}_c \boldsymbol{Q}_{co}(\boldsymbol{A}_c, \boldsymbol{b}_c) = \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) .$$

Now we use relations (3.19b) and then we get

$$\boldsymbol{T}_c[\boldsymbol{b}_c, \boldsymbol{A}_c\boldsymbol{b}_c, \dots, \boldsymbol{A}_c^{n-1}\boldsymbol{b}_c] = \boldsymbol{T}_c[\boldsymbol{T}_c^{-1}\boldsymbol{b}, \boldsymbol{T}_c^{-1}\boldsymbol{A}\boldsymbol{T}_c\boldsymbol{T}_c^{-1}\boldsymbol{b}, \dots, \boldsymbol{T}_c^{-1}\boldsymbol{A}^{n-1}\boldsymbol{T}_c\boldsymbol{T}_c^{-1}\boldsymbol{b}] =$$
$$= [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{b}] = \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}).$$

Supposing that the system is controllable and observable, its transfer function can be determined

$$G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d = \boldsymbol{c}_c^T(s\boldsymbol{I} - \boldsymbol{A}_c)^{-1}\boldsymbol{b}_c + d =$$
$$= \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} + d, \tag{3.22}$$

from which it is evident that the vector $\boldsymbol{c}_c^T$ is given by the coefficients of the transfer function numerator (3.22) [see (3.19b)]. The coefficients in the denominator of the transfer function (3.22) are the coefficients of the characteristic polynomial of the linear dynamic system (3.13) and (3.19) [see (3.17)], i.e.

$$N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = \det(s\boldsymbol{I} - \boldsymbol{A}_c) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 . \tag{3.23}$$

It is very important that, due to the specific structure of the matrix (3.18b), it can be compiled only on the basis of knowledge of the characteristic polynomial coefficients of the original system (3.13) [see (3.23)], i.e., without prior knowledge of the transformed canonical controller form (3.19).

The block diagram of the linear dynamic system in canonical controller form is shown in Fig. 3.5.

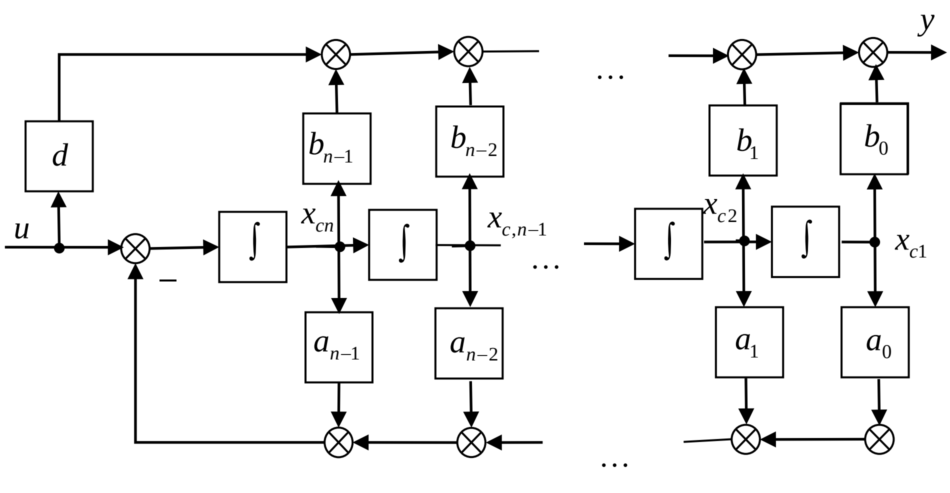


Fig. 3.5 Block diagram of linear dynamic system in canonical controller form

Canonical observer form

For the transformation matrix

$$\boldsymbol{T}_o^{-1} = \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T)\,, \tag{3.24a}$$

where

$$\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T) = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} = \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c)\,, \tag{3.24b}$$

on the basis of the relations (3.15) and (3.16) we get (index t is necessary to replace by the index o) the **canonical** (normal) **observer form**

$$\begin{aligned} \dot{\boldsymbol{x}}_o &= \boldsymbol{A}_o \boldsymbol{x}_o + \boldsymbol{b}_o u, \\ y &= \boldsymbol{c}_o^T \boldsymbol{x}_o + du\,, \end{aligned} \tag{3.25a}$$

where

$$\boldsymbol{A}_o = \boldsymbol{T}_o^{-1}\boldsymbol{A}\boldsymbol{T}_o = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$

$$\boldsymbol{b}_o = \boldsymbol{T}_o^{-1}\boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix}, \;\; \boldsymbol{c}_o^T = \boldsymbol{c}^T \boldsymbol{T}_o = [0, 0, \dots, 0, 1]\,. \tag{3.25b}$$

Also in this case the square matrix (3.24b) has the same form and structure as the matrix (3.18b) and therefore it holds

$$\left|\det \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\right| = \left|\det \boldsymbol{Q}_{ob}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\right| = 1\,. \tag{3.26}$$

From the comparison of relations (3.19) and (3.25) follows that between the canonical controller forms and canonical observer form duality holds

$$\left.\begin{aligned} \dot{\boldsymbol{x}}_c(t) &= \boldsymbol{A}_c \boldsymbol{x}_c(t) + \boldsymbol{b}_c u(t), & \quad \dot{\boldsymbol{x}}_o(t) &= \boldsymbol{A}_o \boldsymbol{x}_o(t) + \boldsymbol{b}_o u(t), \\ y(t) &= \boldsymbol{c}_c^T \boldsymbol{x}_c(t) + du(t), & \quad y(t) &= \boldsymbol{c}_o^T \boldsymbol{x}_o(t) + du(t), \end{aligned}\right\} \tag{3.27}$$

canonical controller form — canonical observer form

where

$$\left.\begin{array}{lll} \boldsymbol{A}_o = \boldsymbol{A}_c^T & \Leftrightarrow & \boldsymbol{A}_c = \boldsymbol{A}_o^T, \\ \boldsymbol{b}_o = \boldsymbol{c}_c & \Leftrightarrow & \boldsymbol{b}_c = \boldsymbol{c}_o, \\ \boldsymbol{c}_o^T = \boldsymbol{b}_c^T & \Leftrightarrow & \boldsymbol{c}_c^T = \boldsymbol{b}_o^T. \end{array}\right\} \tag{3.28}$$

The transfer constant d remains unchanged in all state models.

Both matrices $\boldsymbol{A}_c$ and $\boldsymbol{A}_o = \boldsymbol{A}_c^T$ in both state models (3.27) have the **Frobenius canonical form** characterized in that the first or the last row, or the first or the last column contains the negative coefficients of the characteristic polynomials $N(s)$ for $a_n = 1$. Their characteristic polynomials are the same and they are given by relation

$$\begin{aligned} N(s) &= \det(s\boldsymbol{I} - \boldsymbol{A}) = \det(s\boldsymbol{I} - \boldsymbol{A}_c) = \det(s\boldsymbol{I} - \boldsymbol{A}_o) = \\ &= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = (s - s_1)(s - s_2)\cdots(s - s_n), \end{aligned} \tag{3.29}$$

where s_i are the **eigenvalues** which are the same for matrices $\boldsymbol{A}$, $\boldsymbol{A}_c$ and $\boldsymbol{A}_o = \boldsymbol{A}_c^T$.

The block diagram of the linear dynamic system in canonical observer form is shown in Fig. 3.6.

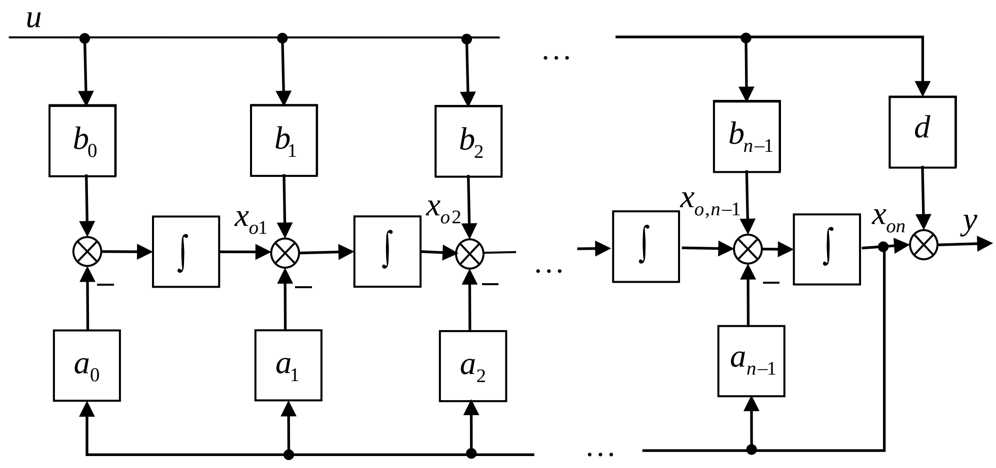


Fig. 3.6 Block diagram of linear dynamic system in canonical controller form

From the above it is clear that the canonical controller form (3.19) and canonical observer form (3.25) we can obtain for the controllable and observable linear dynamic system from the transfer function (3.22) or by the transformation (3.18) and (3.24). Advantageous is the use of a duality between the two canonical forms (3.27) and (3.28).

Canonical diagonal form

Consider the controllable and observable linear dynamic system with the transfer function [see (2.55)]

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} + d\ . \tag{3.30}$$

Assuming that the poles are different from each other, we can write

$$G(s)=\frac{Y(s)}{U(s)}=d+\frac{b_{n-1}s^{n-1}+\ldots+b_1s+b_0}{(s-s_1)(s-s_2)\ldots(s-s_n)}=$$
$$=d+\frac{c_1}{s-s_1}+\frac{c_2}{s-s_2}+\cdots+\frac{c_n}{s-s_n} \quad (3.31)$$

and the state model will

$$\begin{aligned}\dot{x}_{d1}&=s_1x_{d1}+u,\\ \dot{x}_{d2}&=s_2x_{d2}+u,\\ &\vdots\\ \dot{x}_{dn}&=s_nx_{dn}+u,\\ y&=c_1x_{d1}+c_2x_{d2}+\cdots+c_nx_{dn}+du,\end{aligned} \quad (3.32a)$$

or

$$\begin{aligned}\dot{\boldsymbol{x}}_d&=\boldsymbol{A}_d\boldsymbol{x}_d+\boldsymbol{b}_du,\\ y&=\boldsymbol{c}_d^T\boldsymbol{x}_d+du,\end{aligned} \quad (3.32b)$$

where

$$\boldsymbol{A}_d=\begin{bmatrix}s_1 & 0 & \ldots & 0\\ 0 & s_2 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & s_n\end{bmatrix},\ \boldsymbol{b}_d=\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix},\ \boldsymbol{c}_d^T=[c_1,c_2,\ldots,c_n]\,. \quad (3.32c)$$

The state model of the linear dynamic system (3.32) with the matrix $\boldsymbol{A}_d$ in whose diagonal are the poles is called the **canonical diagonal** (modal) **form**.

The block diagram of the linear dynamic system in canonical diagonal form is shown in Fig. 3.7.

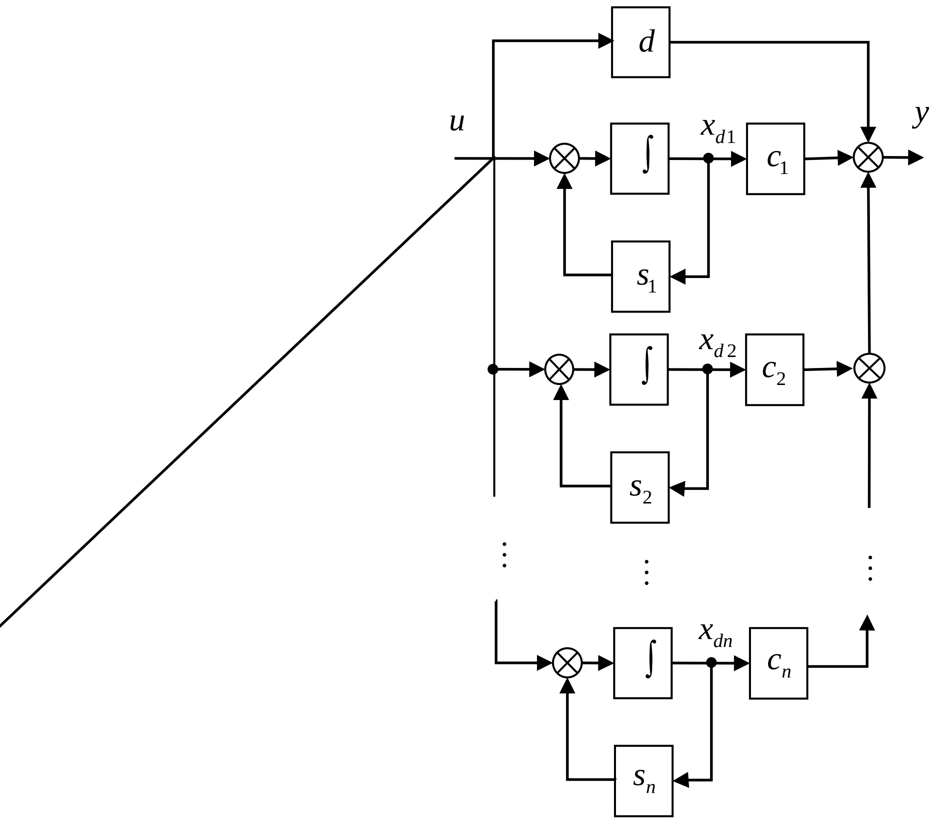


Fig. 3.7 Block diagram of linear dynamic system in canonical diagonal form

State models in canonical diagonal form allow directly to verify their controllability and observability, see Examples 3.3 and 3.4.

Consider now that the transfer function (3.30) has some multiple poles. For simplicity, assume that the multiplicity of pole s_1 is 3 and that the remaining poles are different from each other, i.e.

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} = d + \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{(s-s_1)^3(s-s_4)(s-s_5)\ldots(s-s_n)} = \\ &= d + \frac{c_1}{(s-s_1)^3} + \frac{c_2}{(s-s_1)^2} + \frac{c_3}{s-s_1} + \frac{c_4}{s-s_4} + \cdots + \frac{c_n}{s-s_n}, \end{aligned} \tag{3.33}$$

then the state model will have the form

$$\begin{aligned} \dot{x}_{d1} &= s_1 x_{d1} + x_{d2}, \\ \dot{x}_{d2} &= s_1 x_{d2} + x_{d3}, \\ \dot{x}_{d3} &= s_1 x_{d3} + u, \\ \dot{x}_{d4} &= s_4 x_{d4} + u, \\ &\vdots \\ \dot{x}_{dn} &= s_n x_{dn} + u, \\ y &= c_1 x_{d1} + c_2 x_{d2} + \cdots + c_n x_{dn} + du, \end{aligned} \tag{3.34a}$$

or

$$\begin{aligned} \dot{\boldsymbol{x}}_d &= \boldsymbol{A}_d \boldsymbol{x}_d + \boldsymbol{b}_d u, \\ y &= \boldsymbol{c}_d^T \boldsymbol{x}_d + du, \end{aligned} \tag{3.34b}$$

where

$$\boldsymbol{A}_d = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_2 \end{bmatrix}, \ \boldsymbol{b}_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \ \boldsymbol{c}_d^T = [c_1, c_2, \dots, c_n]\,. \tag{3.34c}$$

The square matrix $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are given by relations

$$\boldsymbol{J}_1 = \begin{bmatrix} s_1 & 1 & 0 \\ 0 & s_1 & 1 \\ 0 & 0 & s_1 \end{bmatrix}, \ \boldsymbol{J}_2 = \begin{bmatrix} s_4 & 0 & \dots & 0 \\ 0 & s_5 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix}. \tag{3.34d}$$

The state model of the linear dynamic system in the form (3.34) is so called **Jordan canonical form** and the square matrices (3.34d) are called **Jordan blocks**.

The block diagram of the linear dynamic system in Jordan canonical form is shown in Fig. 3.8.

The case with multiple real poles can be easily transferred to the case with mutually different poles, e.g. by adding small positive numbers, because the final properties of the dynamic system changes very slightly. E.g. in the transfer function (3.33), we use $s_1 = s_1$, $s_2 = s_1 - \varepsilon$ and $s_3 = s_1 + \varepsilon$, where ε is a very small positive number.

For a transformation of the state model (3.13) on the canonical diagonal or Jordan form it can be also used similarity transformation, but determining the transformation matrix is complex and beyond the scope of this textbook.

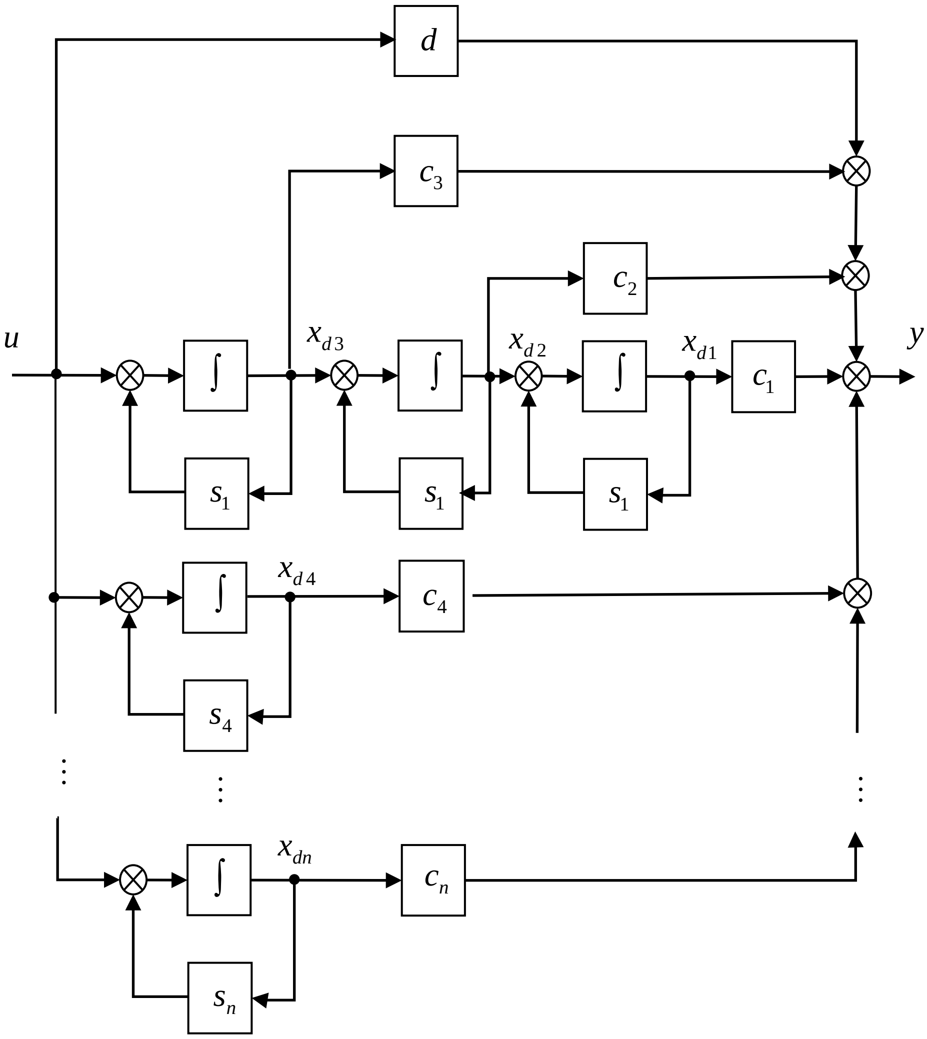


Fig. 3.8 Block diagram of linear dynamic system in Jordan canonical form (3.34)

Example 3.5

The linear dynamic system is described by the state model

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2, \\ \dot{x}_2 &= -x_2 + u, \\ y &= 2x_1 + x_2. \end{aligned} \tag{3.35}$$

The state model (3.35) it is necessary to transform into the above mentioned three canonical forms.

Solution:

For the state model (3.35) holds

$$\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \boldsymbol{c}^T = [2,1], \ d = 0 .$$

We verify the controllability and observability using relations (2.50) and (2.51).

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}] = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}, \quad \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = -2 \neq 0 .$$

The linear dynamic system (3.35) is controllable.

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T \boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}, \quad \det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = 8 \neq 0 .$$

The linear dynamic system (3.35) is observable.

Because the linear dynamical system is controllable and observable a transfer function can be determined. On the basis of the relation (2.54) it can be written

$$N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} s+1 & -2 \\ 0 & s+1 \end{vmatrix} = (s+1)^2 = s^2 + 2s + 1 \Rightarrow s_1 = s_2 = -1 < 0.$$

The linear dynamic system (3.35) is asymptotically stable with the double real pole $s_1 = s_2 = -1$.

$$\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) = \left| \begin{bmatrix} s+1 & -2 \\ 0 & s+1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \right| = \begin{vmatrix} s+1 & -2 \\ 2 & s+2 \end{vmatrix} =$$
$$= (s+1)(s+2) + 4 = s^2 + 3s + 6,$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) - \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \\ = \frac{s+5}{s^2 + 2s + 1} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}. \tag{3.36}$$

Canonical controller form

On the basis of the transfer function (3.36) we can directly write [see (3.19)]

$$\boldsymbol{A}_c = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ \boldsymbol{b}_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \boldsymbol{c}_c^T = [b_0, b_1] = [5,1],$$

i.e.

$$\dot{x}_{c1} = x_{c2},$$
$$\dot{x}_{c2} = -x_{c1} - 2x_{c2} + u,$$
$$y = 5x_{c1} + x_{c2}.$$

Now we use the transformation matrix (3.18):

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c) = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c) = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\boldsymbol{T}_c^{-1} = \frac{\text{adj}\boldsymbol{T}_c}{\det \boldsymbol{T}_c} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix},$$

$$\boldsymbol{A}_c = \boldsymbol{T}_c^{-1}\boldsymbol{A}\boldsymbol{T}_c = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

$$\boldsymbol{b}_c = \boldsymbol{T}_c^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \boldsymbol{c}_c^T = \boldsymbol{c}^T\boldsymbol{T}_c = [2,1]\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = [5,1].$$

We see that we have received the same result. The block diagram of the linear dynamic system (3.35) in canonical controller form is shown in Fig. 3.9.

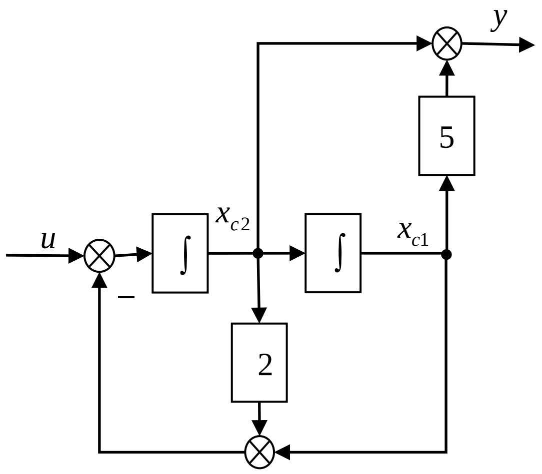


Fig. 3.9 Block diagram of linear dynamic system (3.35) in canonical controller form – Example 3.5

Canonical observer form

On the basis of the transfer function (3.36) we can directly write [see (3.25)]

$$\boldsymbol{A}_o = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \ \boldsymbol{b}_o = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \ \boldsymbol{c}_o^T = [0,1],$$

i.e.

$$\dot{x}_{o1} = -x_{o2} + 5u,$$
$$\dot{x}_{o2} = x_{o1} - 2x_{o2} + u,$$
$$y = x_{o2}.$$

Now we use the transformation matrix (3.24):

$$\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T) = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_o^{-1} = \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix},$$

$$\boldsymbol{T}_o = \frac{\operatorname{adj}\boldsymbol{T}_o^{-1}}{\det \boldsymbol{T}_o^{-1}} = -\frac{1}{8}\begin{bmatrix} 1 & -5 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$\boldsymbol{A}_o = \boldsymbol{T}_o^{-1}\boldsymbol{A}\boldsymbol{T}_o = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix},$$

$$\boldsymbol{b}_o = \boldsymbol{T}_o^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{c}_o^T = \boldsymbol{c}^T\boldsymbol{T}_o = [2,1]\begin{bmatrix} -\frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = [0,1].$$

As in the previous case we have received the same result. It is also clear that among the canonical controller form and canonical observer form the duality holds (3.28).

The block diagram of the linear dynamic system (3.35) in canonical observer form is shown in Fig. 3.10.

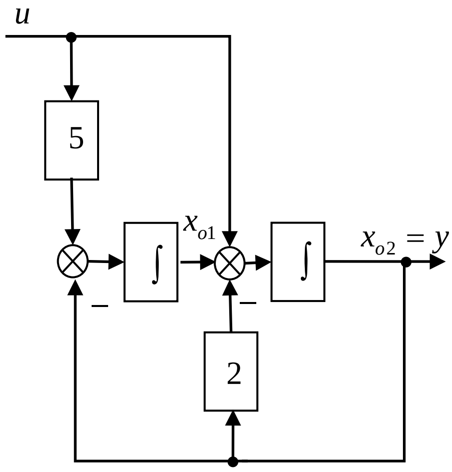


Fig. 3.10 Block diagram of linear dynamic system (3.35) in canonical observer form – Example 3.5

Jordan canonical form

We write the transfer function (3.36) in the form (3.33), i.e.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+5}{(s+1)^2} = \frac{4}{(s+1)^2} + \frac{1}{s+1} = \frac{c_1}{(s+1)^2} + \frac{c_2}{s+1}.$$

On the basis of the relation (3.34) we can directly write

$$\dot{x}_{d1} = -x_{d1} + x_{d2},$$
$$\dot{x}_{d2} = -x_{d2} + u,$$
$$y = 4x_{d1} + x_{d2}.$$

i.e.

$$\boldsymbol{A}_d = \boldsymbol{J} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\ \boldsymbol{b}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \boldsymbol{c}_d^T = [4,1].$$

The block diagram of the linear dynamic system (3.35) in Jordan canonical form is shown in Fig. 3.11.

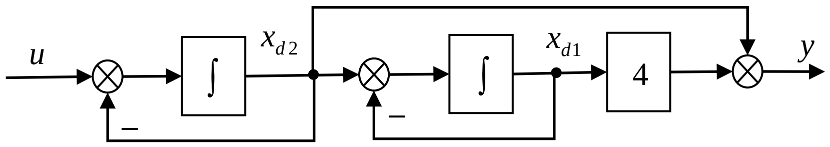


Fig. 3.11 Block diagram of linear dynamic system (3.35) in Jordan canonical form – Example 3.5

3.4 Solution of linear state equations

Consider the linear dynamic system with the state model [see (2.49)]

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t),\quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.37a}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) + du(t). \tag{3.37b}$$

Using Laplace transform and considering the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, we get

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{b}U(s),$$
$$Y(s) = \boldsymbol{c}^T\boldsymbol{X}(s) + dU(s).$$

From the first equation we get

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}_0 + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}U(s)$$

and after substituting into the second equation and modification we receive the transform of the solution

$$Y(s) = \underbrace{\boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}_0}_{\substack{\text{free response =}\\ \text{response to initial conditions}}} + \underbrace{[\boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d]U(s)}_{\substack{\text{forced response =}\\ \text{response to input}}} \tag{3.38}$$

Now we find the solution of the equations (3.37) in the time domain by the method of variation of constants.

Consider that the solution of the equation (3.37) has the form

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{c}(t), \tag{3.39}$$

where

$$\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots, \tag{3.40}$$

this is so called **fundamental matrix** and $\boldsymbol{c}(t)$ is still an unknown vector function.

First, we will show some important properties of the fundamental matrix (3.40):

$$\mathrm{e}^{\boldsymbol{A}0} = \boldsymbol{I}\ , \tag{3.41a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\boldsymbol{A}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots\right) = \\ &= \boldsymbol{A} + \frac{2t}{2!}\boldsymbol{A}^2 + \frac{3t^2}{3!}\boldsymbol{A}^3 + \cdots = \boldsymbol{A}\left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots\right) = \\ &= \left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots\right)\boldsymbol{A} = \\ &= \boldsymbol{A}\,\mathrm{e}^{\boldsymbol{A}t} = \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{A}\,.\end{aligned} \tag{3.41b}$$

$$\begin{aligned}\int \mathrm{e}^{\boldsymbol{A}t}\,\mathrm{d}t &= \int\left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots\right)\mathrm{d}t = \\ &= t\boldsymbol{I} + \frac{t^2}{2\cdot 1!}\boldsymbol{A} + \frac{t^3}{3\cdot 2!}\boldsymbol{A}^2 + \frac{t^4}{4\cdot 3!}\boldsymbol{A}^3 + \cdots = \\ &= \boldsymbol{A}^{-1}\left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots - \boldsymbol{I}\right) = \\ &= \left(\boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots - \boldsymbol{I}\right)\boldsymbol{A}^{-1} = \\ &= \boldsymbol{A}^{-1}\left(\mathrm{e}^{\boldsymbol{A}t} - \boldsymbol{I}\right) = \left(\mathrm{e}^{\boldsymbol{A}t} - \boldsymbol{I}\right)\boldsymbol{A}^{-1}.\end{aligned} \tag{3.41c}$$

After substituting the assumed solution (3.39) into the state equation (3.37a) we get

$$\begin{aligned}&\boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{c}(t) + \mathrm{e}^{\boldsymbol{A}t}\,\dot{\boldsymbol{c}}(t) = \boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{c}(t) + \boldsymbol{b}u(t) \ \Rightarrow \\ &\qquad \dot{\boldsymbol{c}}(t) = \mathrm{e}^{-\boldsymbol{A}t}\,\boldsymbol{b}u(t), \qquad \boldsymbol{c}(0) = \boldsymbol{x}_0\,, \\ &\qquad \boldsymbol{c}(t) = \int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\,\boldsymbol{b}u(\tau)\,\mathrm{d}\tau + \boldsymbol{x}_0.\end{aligned} \tag{3.42}$$

Now we substitute (3.42) into (3.39) and we get

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}_0 + \mathrm{e}^{\boldsymbol{A}t}\left[\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\,u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b} \tag{3.43}$$

and after substitution into the input equation (3.37b) we obtain

$$y(t) = \boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}_0 + \boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\left[\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\,u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b} + du(t)\,, \tag{3.44}$$

where the first part of the solution $\boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}_0$ is the free response = the response to the initial condition and the second part of the solution $\boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\left[\int_0^t \mathrm{e}^{-\boldsymbol{A}t}\,u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b} + du(t)$ is the forced response = the response to the input.

From the comparison of relations (3.44) and (3.38) it follows that the term

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}$$

is the Laplace transform of the fundamental matrix (3.40), i.e.

$$\mathrm{L}\left\{\mathrm{e}^{\boldsymbol{A}t}\right\}=(s\boldsymbol{I}-\boldsymbol{A})^{-1} \Leftrightarrow \mathrm{e}^{\boldsymbol{A}t}=\mathrm{L}^{-1}\left\{(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right\}. \tag{3.45}$$

Now suppose that the input variable of the linear dynamic system has the staircase form (Fig. 3.12)

$$u(t)=u(kT) \quad \text{for} \quad kT \le t<(k+1)T,\ k=1,2,\ldots, \tag{3.46}$$

where kT is the **discrete time**, T – the **sampling period**.

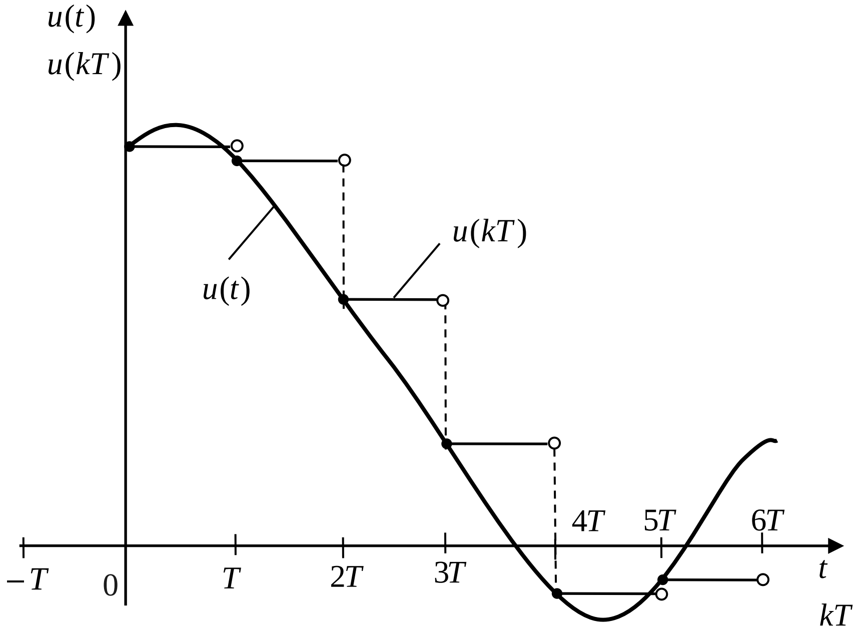


Fig. 3.12 Courses of input variables $u(t)$ and $u(kT)$

On the basis of the relation (3.43) for $t = kT$ and $t = (k + 1)T$ we can write

$$\boldsymbol{x}(kT)=\mathrm{e}^{\boldsymbol{A}kT}\boldsymbol{x}_0+\mathrm{e}^{\boldsymbol{A}kT}\left[\int_0^{kT}\mathrm{e}^{-\boldsymbol{A}\tau}u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b},$$

$$\boldsymbol{x}[(k+1)T]=\mathrm{e}^{\boldsymbol{A}(k+1)T}\boldsymbol{x}_0+\mathrm{e}^{\boldsymbol{A}(k+1)T}\left[\int_0^{(k+1)T}\mathrm{e}^{-\boldsymbol{A}\tau}u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b}=$$

$$=\mathrm{e}^{\boldsymbol{A}T}\underbrace{\left\{\mathrm{e}^{\boldsymbol{A}kT}\boldsymbol{x}_0+\mathrm{e}^{\boldsymbol{A}kT}\left[\int_0^{kT}\mathrm{e}^{-\boldsymbol{A}\tau}u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b}\right\}}_{\boldsymbol{x}(kT)}+\mathrm{e}^{\boldsymbol{A}(k+1)T}\left[\int_{kT}^{(k+1)T}\mathrm{e}^{-\boldsymbol{A}\tau}u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b}=$$

$$=\mathrm{e}^{\boldsymbol{A}T}\boldsymbol{x}(kT)+\left[\int_{kT}^{(k+1)T}\mathrm{e}^{\boldsymbol{A}[(k+1)T-\tau]}\mathrm{d}\tau\right]\boldsymbol{b}u(kT). \tag{3.47}$$

The integral in the last relation can be simplified. We select the new variable

$$v = (k+1)T - \tau \;\Rightarrow\; \mathrm{d}v = -\mathrm{d}\tau,$$
$$\tau = kT \;\Rightarrow\; v = T,\; \tau = (k+1)T \;\Rightarrow\; v = 0$$

and then we can write

$$\int_{kT}^{(k+1)T} \mathrm{e}^{\boldsymbol{A}[(k+1)T-\tau]}\mathrm{d}\tau = -\int_{T}^{0}\mathrm{e}^{\boldsymbol{A}v}\mathrm{d}v = \int_{0}^{T}\mathrm{e}^{\boldsymbol{A}v}\mathrm{d}v .$$

Now the state equation can be written in the form

$$\boldsymbol{x}[(k+1)T] = \mathrm{e}^{\boldsymbol{A}T}\,\boldsymbol{x}(kT) + \left(\int_0^T \mathrm{e}^{\boldsymbol{A}v}\mathrm{d}v\right)\boldsymbol{b}u(kT) . \tag{3.48}$$

On the basis of the relations (3.40) and (3.41) we get

$$\mathrm{e}^{\boldsymbol{A}T} = \boldsymbol{I} + \frac{1}{1!}\boldsymbol{A}T + \frac{1}{2!}(\boldsymbol{A}T)^2 + \cdots = \sum_{i=0}^{\infty}\frac{1}{i!}(\boldsymbol{A}T)^i , \tag{3.49a}$$

$$\int_0^T \mathrm{e}^{\boldsymbol{A}v}\,\mathrm{d}v = T\left[\boldsymbol{I} + \frac{1}{2!}\boldsymbol{A}T + \frac{1}{3!}(\boldsymbol{A}T)^2 + \cdots\right] = T\sum_{i=0}^{\infty}\frac{1}{(i+1)!}(\boldsymbol{A}T)^i . \tag{3.49b}$$

Now the discretized state equation of the linear system (3.37) can be written in the form

$$\boldsymbol{x}[(k+1)T] = \boldsymbol{A}_D\boldsymbol{x}(kT) + \boldsymbol{b}_D u(kT), \tag{3.50a}$$

where

$$\boldsymbol{A}_D = \mathrm{e}^{\boldsymbol{A}T} = \sum_{i=0}^{\infty}\frac{1}{i!}(\boldsymbol{A}T)^i , \tag{3.50b}$$

$$\boldsymbol{b}_D = \left(\int_0^T \mathrm{e}^{\boldsymbol{A}v}\mathrm{d}v\right)\boldsymbol{b} = T\left[\sum_{i=0}^{\infty}\frac{1}{(i+1)!}(\boldsymbol{A}T)^i\right]\boldsymbol{b} . \tag{3.50c}$$

When calculating the matrix $\boldsymbol{A}_D$ and the vector $\boldsymbol{b}_D$ it is suitable to use a numerical method. First we determine the matrix

$$\boldsymbol{D} = T\sum_{i=0}^{\infty}\frac{1}{(i+1)!}(\boldsymbol{A}T)^i , \tag{3.51a}$$

and then we calculate

$$\boldsymbol{A}_D = \boldsymbol{I} + \boldsymbol{A}\boldsymbol{D} , \tag{3.51b}$$

$$\boldsymbol{b}_D = \boldsymbol{D}\boldsymbol{b} . \tag{3.51c}$$

The output equation does not change when it is discretized, and therefore the discretized (discrete) linear dynamic system obtained from the continuous linear dynamic system (3.37) has the form

$$\boldsymbol{x}[(k+1)T] = \boldsymbol{A}_D\boldsymbol{x}(kT) + \boldsymbol{b}_D u(kT),\;\; \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{3.52a}$$

$$y(kT) = \boldsymbol{c}^T\boldsymbol{x}(kT) + du(kT), \tag{3.52b}$$

where the system matrix $\boldsymbol{A}_D$ and the input vector $\boldsymbol{b}_D$ are given by formulas (3.50b) and (3.50c) or (3.51).

The discrete state model (3.52) can be used for a numerical calculation of a response.

Example 3.6

The continuous linear dynamic system is described by the state model

$$\begin{aligned} &\dot{x}_1(t) = -x_1(t) + 2x_2(t), && x_1(0) = x_{10} = 1, \\ &\dot{x}_2(t) = -2x_2(t) + u, && x_2(0) = x_{20} = 2, \\ &y(t) = x_1(t). \end{aligned} \tag{3.53}$$

It is necessary to determine general formulas for a calculating of the response to any input and further it is necessary to determine the step response.

Solution:

For the linear dynamic system (3.53) it can be written

$$\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}, \; \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \boldsymbol{c}^T = [1,0], \; d = 0 .$$

We verify the controllability and observability [see relations (2.50) and (2.51)]

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{Ab}] = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}, \; \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = -2 \neq 0 \;\Rightarrow$$

the linear dynamic system is controllable.

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \; \det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = 2 \;\Rightarrow$$

the linear dynamic system is observable.

Because the linear dynamic system is controllable and observable therefore the state model (3.53) has a minimal form.

Solution in complex variable domain, i.e. on basis of Laplace transform

We determine the transform of the fundamental matrix [see (3.45)]

$$\begin{aligned} \mathrm{L}\left\{\mathrm{e}^{\boldsymbol{A}t}\right\} &= (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} s+1 & -2 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \\ &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 2 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}. \end{aligned} \tag{3.54}$$

In accordance with the relation (3.38) the transform of the response is given by

$$Y(s) = \boldsymbol{c}^T (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}_0 + \boldsymbol{b}U(s)] =$$

$$= [1,0]\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}\left\{\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}U(s)\right\} \Rightarrow$$

$$Y(s) = \left[\frac{1}{s+1}, \frac{2}{(s+1)(s+2)}\right]\left\{\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}U(s)\right\} .$$

$$y(t) = \mathrm{L}^{-1}\{Y(s)\} = \mathrm{L}^{-1}\left\{\left[\frac{1}{s+1}, \frac{2}{(s+1)(s+2)}\right]\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}U(s)\right)\right\} . \tag{3.55}$$

For $u(t) = \eta(t) \Rightarrow U(s) = \dfrac{1}{s}$ we get

$$y(t) = \mathrm{L}^{-1}\left\{\frac{s^2 + 6s + 2}{s(s+1)(s+2)}\right\} = 1 + 3\mathrm{e}^{-t} - 3\mathrm{e}^{-2t} . \tag{3.56}$$

The course of the step response is in Fig. 3.13.

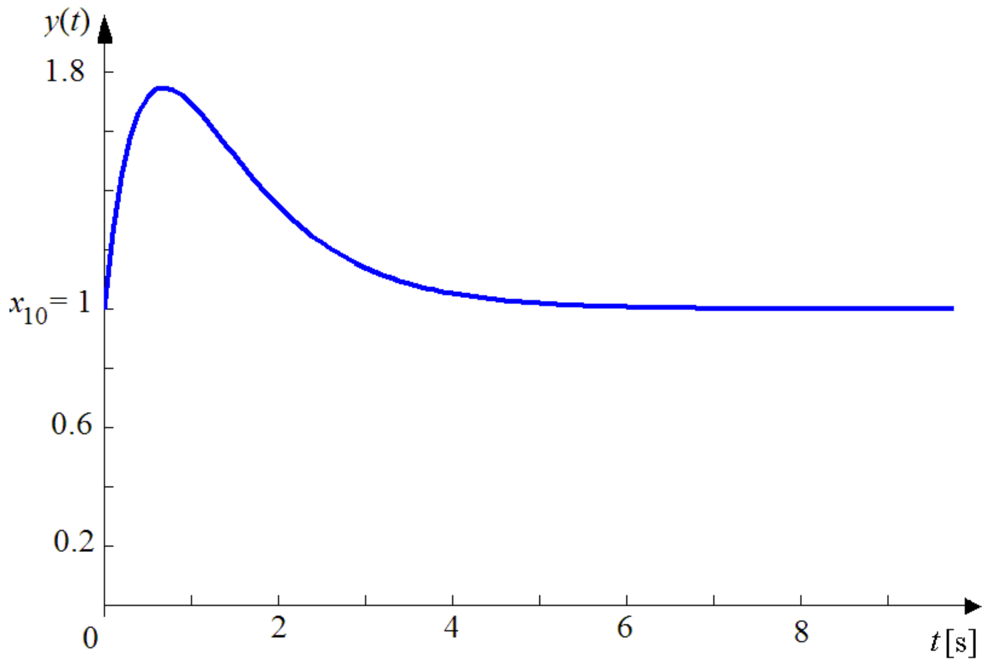


Fig. 3.13 Step response of linear dynamic system (3.53) – Example 3.6

Solution in time domain

In accordance with the relation (3.44) we can write

$$y(t) = \boldsymbol{c}^T \mathrm{e}^{\boldsymbol{A}t}\left\{\boldsymbol{x}_0 + \left[\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau} u(\tau)\,\mathrm{d}\tau\right]\boldsymbol{b}\right\}. \tag{3.57}$$

From the relation (3.54) we determine the fundamental matrix

$$\mathrm{e}^{\boldsymbol{A}t}=\mathrm{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+2)}\\ 0 & \dfrac{1}{s+2}\end{bmatrix}\right\}=\begin{bmatrix}\mathrm{e}^{-t} & 2\,\mathrm{e}^{-t}-2\,\mathrm{e}^{-2t}\\ 0 & \mathrm{e}^{-2t}\end{bmatrix}. \tag{3.58}$$

The fundamental matrix (3.58) we substitute into (3.57) and after modification we get

$$y(t)=[\mathrm{e}^{-t},\ 2\,\mathrm{e}^{-t}-2\,\mathrm{e}^{-2t}]\left\{\begin{bmatrix}1\\2\end{bmatrix}+\left(\int_0^t\begin{bmatrix}\mathrm{e}^{\tau} & 2\,\mathrm{e}^{\tau}-2\,\mathrm{e}^{2\tau}\\ 0 & \mathrm{e}^{2\tau}\end{bmatrix}u(\tau)\,\mathrm{d}\,\tau\right)\boldsymbol{b}\right\}. \tag{3.59}$$

We see that the general formula is rather complicated for the calculation of the response to any input in the time domain.

Now consider the input in the form of the unit step $u(t)=\eta(t)=1$ for $t\geq 0$.

First we calculate the expression with integral

$$\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\mathrm{d}\,\tau=\int_0^t\begin{bmatrix}\mathrm{e}^{\tau} & 2\,\mathrm{e}^{\tau}-2\,\mathrm{e}^{2\tau}\\ 0 & \mathrm{e}^{2\tau}\end{bmatrix}\mathrm{d}\,\tau=\begin{bmatrix}\mathrm{e}^{t}-1 & 2\,\mathrm{e}^{t}-\mathrm{e}^{2t}-1\\ 0 & \dfrac{1}{2}\mathrm{e}^{2t}-\dfrac{1}{2}\end{bmatrix},$$

$$\left[\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\mathrm{d}\,\tau\right]\boldsymbol{b}=\begin{bmatrix}\mathrm{e}^{t}-1 & 2\,\mathrm{e}^{t}-\mathrm{e}^{2t}-1\\ 0 & \dfrac{1}{2}\mathrm{e}^{2t}-\dfrac{1}{2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\,\mathrm{e}^{t}-\mathrm{e}^{2t}-1\\ \dfrac{1}{2}\mathrm{e}^{2t}-\dfrac{1}{2}\end{bmatrix}. \tag{3.60}$$

After substitution into (3.59) and modification we get

$$y(t)=[\mathrm{e}^{-t},\ 2\,\mathrm{e}^{-t}-2\,\mathrm{e}^{-2t}]\left\{\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\,\mathrm{e}^{t}-\mathrm{e}^{2t}-1\\ \dfrac{1}{2}\mathrm{e}^{2t}-\dfrac{1}{2}\end{bmatrix}\right\}=1+3\,\mathrm{e}^{-t}-3\,\mathrm{e}^{-2t}.$$

We received the same result as in the previous case.

Discretization of continouous linear dynamic system

For the discretization we use first analytical relations (3.50b) and (3.50c) and later numerical relations (3.51) for i = 0, 1, 2, 3. The sampling period is chosen e.g. T = 0.1.

On the basis of relations (3.50b) and (3.58) we can write (we consider 5 decimal places).

$$\boldsymbol{A}_D=\mathrm{e}^{\boldsymbol{A}T}=\begin{bmatrix}\mathrm{e}^{-T} & 2\,\mathrm{e}^{-T}-2\,\mathrm{e}^{-2T}\\ 0 & \mathrm{e}^{-2T}\end{bmatrix}\doteq\begin{bmatrix}0{,}90484 & 0{,}17221\\ 0 & 0{,}81873\end{bmatrix}.$$

Similarly on the basis of relationships (3.50c) and (3.58) we get

$$\boldsymbol{b}_D=\left[\int_0^T \mathrm{e}^{\boldsymbol{A}v}\mathrm{d}\,v\right]\boldsymbol{b}=\left\{\int_0^T\begin{bmatrix}\mathrm{e}^{-\tau} & 2\,\mathrm{e}^{-\tau}-2\,\mathrm{e}^{-2\tau}\\ 0 & \mathrm{e}^{-2\tau}\end{bmatrix}\mathrm{d}\,\tau\right\}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1-2\,\mathrm{e}^{-T}+\mathrm{e}^{-2T}\\ \dfrac{1}{2}-\dfrac{1}{2}\mathrm{e}^{-2T}\end{bmatrix}\doteq$$

$$\doteq\begin{bmatrix}0{,}00906\\ 0{,}09063\end{bmatrix}.$$

Now we use the relations (3.51) for i = 0, 1, 2, 3:

$$\boldsymbol{D} = T\left[\boldsymbol{I} + \frac{1}{2!}\boldsymbol{A}T + \frac{1}{3!}(\boldsymbol{A}T)^2 + \frac{1}{4!}(\boldsymbol{A}T)^3\right] \doteq \frac{1}{144}\begin{bmatrix} 13{,}7034 & 1{,}3044 \\ 0 & 13{,}0512 \end{bmatrix},$$

$$\boldsymbol{A}_D = \boldsymbol{I} + \boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} 0{,}90484 & 0{,}17221 \\ 0 & 0{,}81873 \end{bmatrix},$$

$$\boldsymbol{b}_D = \boldsymbol{D}\boldsymbol{b} \doteq \begin{bmatrix} 0{,}00906 \\ 0{,}09063 \end{bmatrix}.$$

We see that after rounding in both cases we get the same results.

4 STATE SPACE CONTROL

The chapter briefly describes the design of a state controller and observer for the SISO linear dynamic system.

4.1 State space controller

Development of a **state space control** is associated with the development of aeronautics and astronautics. It allows to control very complex and unstable systems, where classical control with one and two degrees of freedom controllers does not give satisfactory results.

Consider the SISO controlled linear dynamic system (in state space methods the name “controlled system” is most often used instead of the controlled plant)

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{4.1a}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \tag{4.1b}$$

which is controllable, observable [see (2.50) and (2.51)] and strongly physically realizable ($d = 0$). Its characteristic polynomial has the form

$$\begin{aligned} N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = \\ = (s - s_1)(s - s_2)\ldots(s - s_n), \end{aligned} \tag{4.2}$$

where s_1, s_2,…, s_n are the system poles.

The task of the **state space controller** (state feedback, feedback controller) represented by the vector (Fig. 4.1)

$$\boldsymbol{k} = [k_1, k_2, \ldots, k_n]^T, \tag{4.3}$$

is to ensure for the closed-loop control system its characteristic polynomial

$$\begin{aligned} N_{kw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w) = s^n + a_{n-1}^w s^{n-1} + \ldots + a_1^w s + a_0^w = \\ = (s - s_1^w)(s - s_2^w)\ldots(s - s_n^w) \end{aligned} \tag{4.4}$$

with given poles $s_1^w, s_2^w, \ldots, s_n^w$ (see Appendix E).

A feedback control by using a state controller (4.3) ensures the characteristic polynomial of the closed-loop control system (4.4) with the desired poles is often called a **modal control**. The poles determine so called **modes**, i.e. the characteristic (free) moves of a closed-loop control system.

The closed-loop control system with the state space controller in accordance with Fig 4.1 may be described by the state model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_w\boldsymbol{x}(t) + \boldsymbol{b}w'(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{4.5a}$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \tag{4.5b}$$

where the system matrix of the closed-loop control system is given (see Fig. 4.1b)

$$\boldsymbol{A}_w = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T. \tag{4.6}$$

The vector $\boldsymbol{k}$ of the state space controller can be obtained by comparing the coefficients of the control system characteristic polynomial $N_k(s) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T)]$ with the corresponding coefficients of the desired control system characteristic polynomial $N_{kw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w)$ at the same powers of complex variable s. In such a way the system of n linear equations is obtained for n unknown components k_i of the vector $\boldsymbol{k}$. For large n, this procedure is demanding.

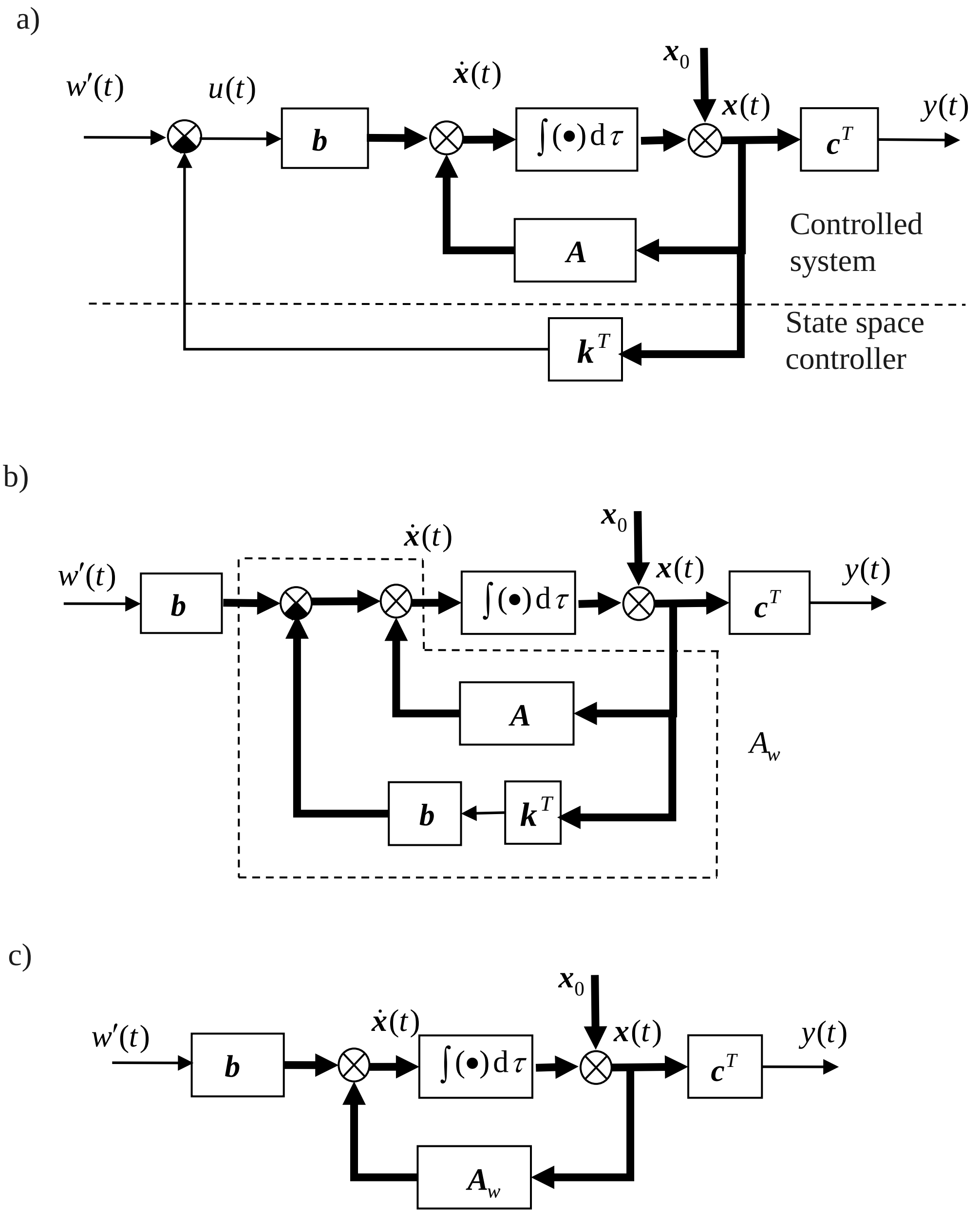


Fig. 4.1 Block diagram of the control system with a state space controller without input correction: a) original, b) modified, c) resultant

The dependence between output $y_w(t)$ and input $w'(t)$ in the steady state ($t \to \infty$) can be determined on the basis of (2.53), i.e.

$$y = \lim_{s\to 0}[\boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A}_w)^{-1}\boldsymbol{b}]\,w' \;\Rightarrow$$

$$y = -\boldsymbol{c}^T \boldsymbol{A}_w{}^{-1}\boldsymbol{b}\,w' \,. \tag{4.7}$$

In order to in the steady state the equality

$$y = w \tag{4.8}$$

holds, the correction

$$k_w = -\frac{1}{\boldsymbol{c}^T \boldsymbol{A}_w^{-1}\boldsymbol{b}}\,. \tag{4.9}$$

in the input must be placed (Fig. 4.2)

The state space controller design is easy for the state space model of the controlled system in the canonical controller form (3.19).

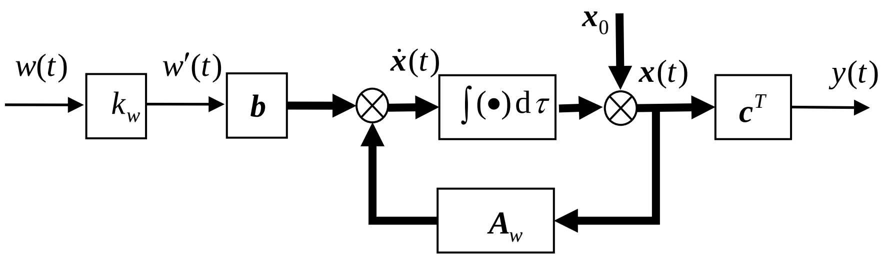


Fig. 4.2 Block diagram of the control system with a state space controller

Consider that the matrices $\boldsymbol{A}$ and $\boldsymbol{A}_w$ are transformed into canonical controller forms in accordance with the relations (3.18), (3.19), then equation (4.6) can be written in the canonical controller form

$$\boldsymbol{A}_{wc} = \boldsymbol{A}_c - \boldsymbol{b}_c\boldsymbol{k}_c^T \,. \tag{4.10a}$$

i.e.

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0^w & -a_1^w & -a_2^w & \dots & -a_{n-1}^w \end{bmatrix} = \tag{4.10b}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [k_{c1}, k_{c2}, \dots, k_{cn}].$$

We can see that the equalities hold

$$-a_{i-1}^w = -a_{i-1} - k_{ci} \;\Rightarrow$$

$$k_{ci} = a_{i-1}^{w} - a_{i-1} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.11}$$

The last equalities can be written in the vector form

$$\boldsymbol{k}_c = \boldsymbol{a}^w - \boldsymbol{a}\,, \tag{4.12}$$

where

$$\boldsymbol{a}^w = [a_0^w, a_1^w, \ldots, a_{n-1}^w]^T\,, \tag{4.13a}$$

$$\boldsymbol{a} = [a_0, a_1, \ldots, a_{n-1}]^T \tag{4.13b}$$

are the vectors of the coefficients of the characteristic polynomials $N_w(s)$ and $N(s)$ [see (4.4) and (4.2)].

We have received the vector $\boldsymbol{k}_c$ of the feedback state space controller in the canonical controller form, and we must therefore transform it back for the original controlled system (4.1). We can write

$$\left.\begin{aligned} \boldsymbol{k}_c^T \boldsymbol{x}_c &= \boldsymbol{k}^T \boldsymbol{x} \\ \boldsymbol{x}_c &= \boldsymbol{T}_c^{-1} \boldsymbol{x} \end{aligned}\right\} \Rightarrow \boldsymbol{k}^T = \boldsymbol{k}_c^T \boldsymbol{T}_c^{-1} \Rightarrow$$

$$\boldsymbol{k}^T = (\boldsymbol{a}^w - \boldsymbol{a})^T \boldsymbol{T}_c^{-1}\,, \tag{4.14}$$

where the transformation matrix $\boldsymbol{T}_c$ is given by the relations [see (3.18)]

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b})\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c)\,, \tag{4.15a}$$

$$\boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b}]\,, \tag{4.15b}$$

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c) = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{4.15c}$$

The relation (4.14) is sometimes called the **Bass-Gura formula**.

For the direct calculation of the feedback vector $\boldsymbol{k}^T$ is often used **Ackermann´s formula** (see Appendix D)

$$\begin{aligned} \boldsymbol{k}^T &= [0,0,\ldots,0,1]\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}, \boldsymbol{b}) N_{kw}(\boldsymbol{A}) = \\ &= [0,0,\ldots,0,1]\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}, \boldsymbol{b})[\boldsymbol{A}^n + a_{n-1}^w \boldsymbol{A}^{n-1} + \ldots + a_1^w \boldsymbol{A} + a_0^w \boldsymbol{I}]. \end{aligned} \tag{4.16}$$

Procedure:

1. Check the controllability and the observability of the controlled system (plant) [relations (2.50) and (2.51)].
2. Formulate the requirements for the control performance and express it by the desired pole placement of the control system (see Appendix E).
3. Determine the coefficients of the characteristic polynomials $N(s)$ and $N_{kw}(s)$

[relations (4.2) and (4.4)].

4. Compare the coefficients of the control system characteristic polynomial $N_k(s) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T)]$ with the corresponding coefficients of the desired control system characteristic polynomial $N_{kw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w)$ at the same powers of complex variable s and solve the system of n linear equations for n unknown components of the vector $\boldsymbol{k}$. In the case of high n use the transformation matrix (4.15) and the formula (4.14) or the Aclermann's formula (4.16).
5. On the basis of the relation (4.9) determine the input correction k_w.
6. Verify the received control performance by a simulation.

Example 4.1

For the SISO linear dynamic controlled system (plant) with the state model

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - u, \\ \dot{x}_2 &= x_1 + x_2 + u, \\ y &= x_1 \end{aligned} \tag{4.17}$$

it is necessary to design the state space controller which ensures for the closed-loop control system the poles

$$s_1^w = s_2^w = -1 .$$

Solution:

It is obvious that for the controlled system (4.17) the relations hold

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \boldsymbol{c}^T = [1, 0], \ d = 0 .$$

First we verify on the bases of the relations (2.50) and (2.51) the controllability and the observability.

$$\boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}] = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = -2 \neq 0 \ \Rightarrow$$

The controlled linear dynamical system (4.17) is controllable.

$$\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T \boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \det \boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = 1 \neq 0 \ \Rightarrow$$

The controlled linear dynamical system (4.17) is observable.

Because the controlled linear dynamic system is controllable and observable we can determine on the basis of the relation (2.54) its transfer function

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} s+1 & -1 \\ -1 & s-1 \end{vmatrix} = s^2 - 2 ,$$

$$\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) = \left\| \begin{bmatrix} s+1 & -1 \\ -1 & s-1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\| = s^2 - s ,$$

$$G_{uy}(s)=\frac{Y(s)}{U(s)}=\frac{\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{bc}^T)-\det(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{-s+2}{s^2-2}. \tag{4.18}$$

The controlled linear dynamic system described by the state model (4.17) or the transfer function (4.18) is unstable with the poles $s_{1,2}=\pm\sqrt{2}$ and it is also with a nonminimum phase and an unstable zero. In this case a using a conventional controller and its tuning is not only very difficult but also inappropriate.

The coefficients of polynomials in the denominator and the numerator of the transfer function (4.18) are:

$$\begin{aligned}&a_0=-2, a_1=0 \ \Rightarrow \boldsymbol{a}=[a_0,a_1]^T=[-2,0]^T,\\&b_0=2, b_1=-1.\end{aligned} \tag{4.19}$$

The desired characteristic polynomial of the closed-loop control system (4.4) has the form

$$\begin{aligned}N_{kw}(s)&=\det(s\boldsymbol{I}-\boldsymbol{A}_w)=s^2+a_1^w s+a_0^w=(s-s_1^w)(s-s_2^w)=\\&=s^2+2s+1.\end{aligned} \tag{4.20}$$

The coefficients of the desired characteristic polynomial $N_{kw}(s)$ are:

$$a_0^w=1, a_1^w=2 \ \Rightarrow \ \boldsymbol{a}^w=\left[a_0^w,a_1^w\right]^T=[1,2]^T. \tag{4.21}$$

Method of comparison of coefficient

On the basis of the relation (4.6) we determine the closed-loop control system matrix

$$\boldsymbol{A}_w=\boldsymbol{A}-\boldsymbol{bk}^T=\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}-1\\ 1\end{bmatrix}[k_1,k_2]=\begin{bmatrix}-1+k_1 & 1+k_2\\ 1-k_1 & 1-k_2\end{bmatrix}.$$

The characteristic polynomial of the closed-loop control system is

$$\begin{aligned}N_k(s)&=\det(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{bk^T})=\begin{vmatrix}s+1-k_1 & -1-k_2\\ -1+k_1 & s-1+k_2\end{vmatrix}=\\&=s^2+(k_2-k_1)s+2k_1-2.\end{aligned} \tag{4.22}$$

Now we compare the coefficients of polynomials (4.22) and (4.20), i.e.

$$\left.\begin{aligned}k_2-k_1&=2\\ 2k_1-2&=1\end{aligned}\right\}\Rightarrow \left.\begin{aligned}k_1&=\frac{3}{2}\\ k_2&=\frac{7}{2}\end{aligned}\right\}\Rightarrow \boldsymbol{k^T}=\left[\frac{3}{2},\frac{7}{2}\right] \tag{4.23}$$

Using the relation (4.9) we determine the input filter (correction)

$$\boldsymbol{A}_w=\begin{bmatrix}-1+k_1 & 1+k_2\\ 1-k_1 & 1-k_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{9}{2}\\ -\frac{1}{2} & -\frac{5}{2}\end{bmatrix},$$

$$\boldsymbol{A}_w^{-1} = \frac{\operatorname{adj} \boldsymbol{A}_w}{\det \boldsymbol{A}_w} = \frac{1}{-\frac{5}{4}+\frac{9}{4}}\begin{bmatrix} -\frac{5}{2} & -\frac{9}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & -\frac{9}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\frac{1}{k_w} = -\boldsymbol{c}^T \boldsymbol{A}_w^{-1} \boldsymbol{b} = -[1,0]\begin{bmatrix} -\frac{5}{2} & -\frac{9}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \;\Rightarrow\; k_w = \frac{1}{2}.$$

The block diagram of the control system with the designed state space controller (4.23) is shown in Fig. 4.3 and its step response for zero initial conditions ($\boldsymbol{x}_0 = \boldsymbol{0}$) is shown in Fig. 4.4. The initial undershoot is due to the unstable zero $s_1^0 = 2$ [see (4.18)].

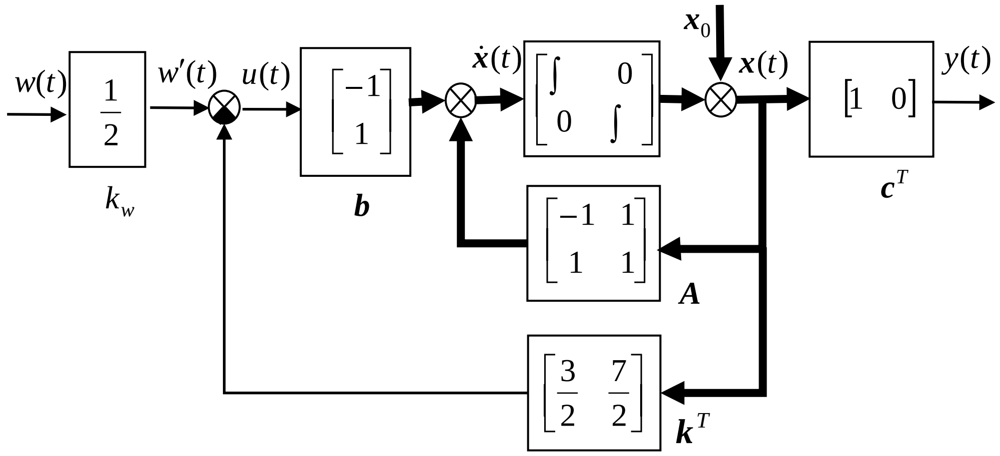


Fig. 4.3 Block diagram of control system with state space controller – Example 4.1

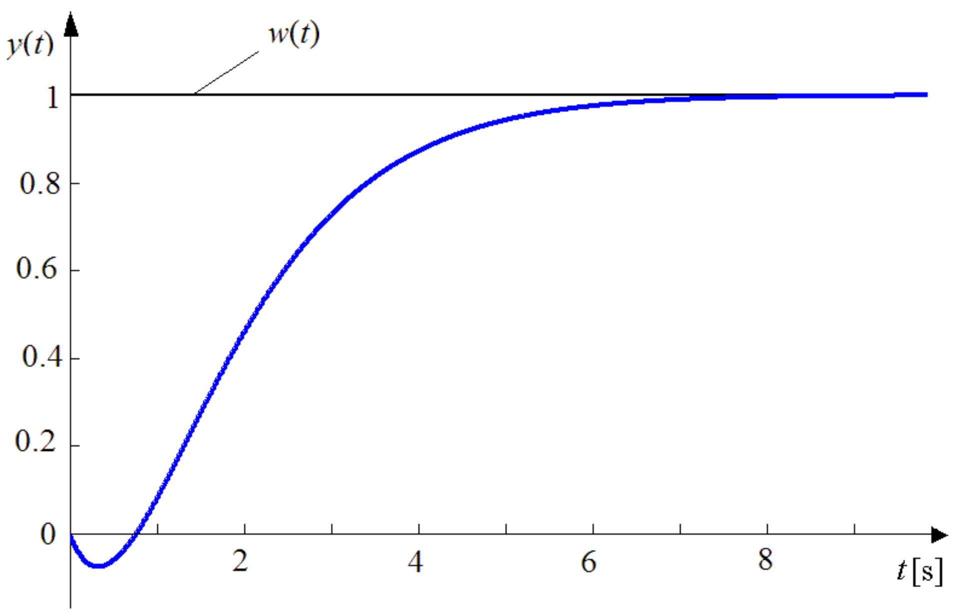


Fig. 4.4 Step response of control system – Example 4.1

Method of transformation

In accordance with the relation (4.15) we can write

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c,\boldsymbol{b}_c) = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{T}_c^{-1} = \frac{\operatorname{adj}\boldsymbol{T}_c}{\det \boldsymbol{T}_c} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

Now we use the relation (4.14) for (4.19) and (4.21)

$$\boldsymbol{k}^T = \left(\boldsymbol{a}^w - \boldsymbol{a}\right)^T \boldsymbol{T}_c^{-1} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right\}^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{7}{2} \end{bmatrix}.$$

We received the same result, see (4.23).

Ackermann's formula

We will use the Ackermann's formula (4.16) and we get

$$\boldsymbol{k}^T = [0 \quad 1]\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A},\boldsymbol{b})[\boldsymbol{A}^2 + 2\boldsymbol{A} + \boldsymbol{I}],$$

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A},\boldsymbol{b}) = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{\operatorname{adj}\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})}{\det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})} = \frac{1}{-2}\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{A}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

After substitution and modification we obtain the same result as in the two previous cases, i.e.

$$\boldsymbol{k}^T = [0 \quad 1]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\left\{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} \frac{3}{2} & \frac{7}{2} \end{bmatrix}.$$

It is obvious that for a higher orders a digital computer is suitable.

Example 4.2

For the SISO linear dynamic controlled system (plant)

$$\begin{aligned} \dot{x}_1 &= -x_1 - 4x_3 + 2u, \\ \dot{x}_2 &= 2x_1 - 2x_2 - 2x_3 + u, \\ \dot{x}_3 &= -4x_3 - 2u, \\ y &= -2x_1 + 4x_2 + x_3 \end{aligned}$$

it is necessary to design the state space controller which ensures for the closed-loop control system the poles

$$s_1^w = s_2^w = s_3^w = -2\,.$$

Solution:

It is obvious that for the controlled system the relations hold

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} -1 & 0 & -4 \\ 2 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \ \boldsymbol{c}^T = \begin{bmatrix} -2 & 4 & 1 \end{bmatrix}.$$

Controllability verification:

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}] = \begin{bmatrix} 2 & 6 & -38 \\ 1 & 6 & -16 \\ -2 & 8 & -32 \end{bmatrix},$$

$\det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = -504 \neq 0 \ \Rightarrow$ The controlled system is controllable.

Observability verification:

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \boldsymbol{c}^T\boldsymbol{A}^2 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 1 \\ 10 & -8 & -4 \\ -26 & 16 & -8 \end{bmatrix},$$

$\det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = 432 \neq 0 \ \Rightarrow$ The controlled system is observable.

From the controlled system transfer function

$$G_{uy}(s) = \frac{Y(s)}{U(s)} = \frac{\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^T) - \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{-2s^2 + 6s + 92}{s^3 + 7s^2 + 14s + 8}$$

it follows: $a_0 = 8$, $a_1 = 14$, $a_2 = 7$, $a_3 = 1$, $b_0 = 92$, $b_1 = 6$, $b_2 = -2$, i.e.

$$\boldsymbol{a} = \begin{bmatrix} 8, & 14, & 7 \end{bmatrix}^T, \ \boldsymbol{c}_c = \begin{bmatrix} 92, & 6, & -2 \end{bmatrix}^T.$$

The desired control system characteristic polynomial has the form

$$N_{kw}(s) = (s+2)^3 = s^3 + 6s^2 + 12s + 8,$$

and therefore the vector of its coefficients is

$$\boldsymbol{a}^w = \begin{bmatrix} 8, & 12, & 6 \end{bmatrix}^T.$$

Method of transformation

The transformation matrix (4.15) has the form

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c,\boldsymbol{b}_c) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}] \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 20 & 2 \\ 40 & 13 & 1 \\ -4 & -6 & -2 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{T}_c^{-1} = \begin{bmatrix} -\frac{5}{126} & \frac{1}{18} & -\frac{1}{84} \\ \frac{19}{126} & -\frac{1}{9} & \frac{2}{21} \\ -\frac{47}{126} & \frac{2}{9} & -\frac{16}{21} \end{bmatrix} .$$

On the basis of the relations (4.14) there is obtained

$$\boldsymbol{k}^T = \left(\boldsymbol{a}^w - \boldsymbol{a}\right)^T \boldsymbol{T}_c^{-1} = \begin{bmatrix} \frac{1}{14}, & 0, & \frac{4}{7} \end{bmatrix}.$$

Ackermann’s formula

On the basis of the Ackermann's formula (4.16) we can write:

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A},\boldsymbol{b}) = \begin{bmatrix} 2 & 6 & -38 \\ 1 & 6 & -16 \\ -2 & 8 & -32 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{8}{63} & \frac{2}{9} & -\frac{11}{42} \\ -\frac{8}{63} & \frac{5}{18} & \frac{1}{84} \\ -\frac{5}{126} & \frac{1}{18} & -\frac{1}{84} \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & -4 \\ 2 & -2 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \boldsymbol{A}^2 = \begin{bmatrix} 1 & 0 & 20 \\ -6 & 4 & 4 \\ 0 & 0 & 16 \end{bmatrix}, \quad \boldsymbol{A}^3 = \begin{bmatrix} -1 & 0 & -84 \\ 14 & -8 & 0 \\ 0 & 0 & -64 \end{bmatrix},$$

$$N_{kw}(\boldsymbol{A}) = \boldsymbol{A}^3 + 6\boldsymbol{A}^2 + 12\boldsymbol{A} + 8\boldsymbol{I} = \begin{bmatrix} 1 & 0 & -12 \\ 2 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

$$\boldsymbol{k}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A},\boldsymbol{b}) N_{kw}(\boldsymbol{A}) = \begin{bmatrix} \frac{1}{14} & 0 & \frac{4}{7} \end{bmatrix}.$$

We received the same result.

The state model of the closed-loop control system without the input correction (filter) will be in the form

$$\boldsymbol{A}_w = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T = \begin{bmatrix} -\frac{8}{7} & 0 & -\frac{36}{7} \\ \frac{27}{14} & -2 & -\frac{18}{7} \\ \frac{1}{7} & 0 & -\frac{20}{7} \end{bmatrix},$$

$$\boldsymbol{b} = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}^T, \quad \boldsymbol{c} = \begin{bmatrix} -2 & 4 & 1 \end{bmatrix}^T,$$

i.e.

$$\dot{x}_1 = -\frac{8}{7}x_1 - \frac{36}{7}x_3 + 2w',$$
$$\dot{x}_2 = \frac{27}{14}x_1 - 2x_2 - \frac{18}{7}x_3 + w',$$
$$\dot{x}_3 = \frac{1}{7}x_1 - \frac{20}{7}x_3 - 2w',$$
$$y = -2x_1 + 4x_2 + x_3.$$

The input correction is given by the relation (4.9)

$$k_w = -\frac{1}{\boldsymbol{c}^T \boldsymbol{A}_w^{-1} \boldsymbol{b}} = \frac{2}{23}.$$

and the corresponding state model of the control system with the input correction has the form

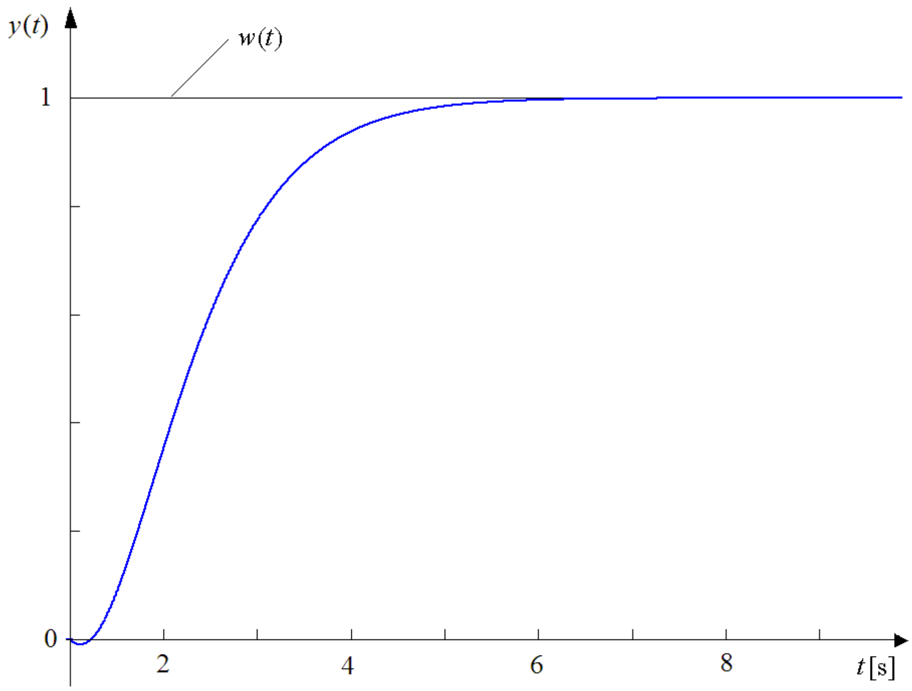


Fig. 4.5 Step response of control system with state space controller and input correction – Example 4.2

$$\dot{x}_1 = -\frac{8}{7}x_1 - \frac{36}{7}x_3 + \frac{4}{23}w,$$
$$\dot{x}_2 = \frac{27}{14}x_1 - 2x_2 - \frac{18}{7}x_3 + \frac{2}{23}w,$$
$$\dot{x}_3 = \frac{1}{7}x_1 - \frac{20}{7}x_3 - \frac{4}{23}w,$$
$$y = -2x_1 + 4x_2 + x_3.$$

The step response of the control system with the state space controller and the input correction is shown in Fig. 4.5. The initial undershoot is caused by the unstable zero ($s_1^0 \doteq 8.446$).

4.2 State observer

The state variables in real dynamic system cannot often be measured due to their unavailability or high measuring noise and costs. In these cases it is necessary to use the **state observer** (estimator).

We will focus on the design of the **Luenberger asymptotic full order observer** (further only the observer), i.e. such the observer which estimates the state variables $\hat{\boldsymbol{x}}(t)$ which are asymptotically approaching the real state variables $\boldsymbol{x}(t)$.

Consider the SISO linear dynamical system (4.1), which is controllable, observable and strongly physically realizable with the characteristic polynomial (4.2).

For this linear dynamic system the Luenberger observer has the form (Fig. 4.6)

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{A}_l \hat{\boldsymbol{x}}(t) + \boldsymbol{b}_l u(t) + \boldsymbol{l} y(t), \quad \hat{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_0, \\ \hat{y}(t) &= \boldsymbol{c}_l^T \hat{\boldsymbol{x}}(t), \end{aligned} \tag{4.24}$$

where $\boldsymbol{A}_l$ is the square observer matrix of order n [(n×n)], $\boldsymbol{b}_l$ – the vector of observer input of the dimension n, $\boldsymbol{c}_l$ – the vector of observer output of the dimension n, $\boldsymbol{l}$ – the vector of **Luenberger observer gain** (correction) of the dimension n, by „^“ are marked the asymptotic estimates of the corresponding variables.

After the definition of the state error vector $\boldsymbol{\varepsilon}(t)$ by the relation

$$\boldsymbol{\varepsilon}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \tag{4.25}$$

and considering the relations (4.1) and (4.24) we get

$$\dot{\boldsymbol{\varepsilon}}(t) = (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^T)\boldsymbol{x}(t) - \boldsymbol{A}_l \hat{\boldsymbol{x}}(t) + (\boldsymbol{b} - \boldsymbol{b}_l)u(t)\,. \tag{4.26}$$

It is clear that the state error vector $\boldsymbol{\varepsilon}(t)$ should not depend on the input variable $u(t)$ and the estimate $\hat{y}(t)$ for the real state $\boldsymbol{x}(t)$ should be $\boldsymbol{c}^T\boldsymbol{x}(t)$, and therefore it must hold

$$\boldsymbol{b}_l = \boldsymbol{b}, \;\; \boldsymbol{c}_l = \boldsymbol{c}\,. \tag{4.27}$$

If we choose

$$\boldsymbol{A}_l = \boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^T \tag{4.28}$$

then for the assumption (4.27) the linear differential equation

$$\dot{\boldsymbol{\varepsilon}}(t) = \boldsymbol{A}_l \boldsymbol{\varepsilon}(t), \quad \boldsymbol{\varepsilon}_0 = \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0 \tag{4.29}$$

is obtained which describes the time course of the state error $\boldsymbol{\varepsilon}(t)$. The initial estimate $\hat{\boldsymbol{x}}_0$ is supposed to be zero in most cases.

It is clear that for the asymptotic state estimate $\hat{\boldsymbol{x}}(t)$ it must hold

$$t \to \infty \;\Rightarrow\; \hat{\boldsymbol{x}}(t) \to \boldsymbol{x}(t) \;\Rightarrow\; \boldsymbol{\varepsilon}(t) \to \boldsymbol{0}\,, \tag{4.30}$$

i.e. the linear differential equation (4.29) must be asymptotically stable.

It is obvious that in order for the state estimate $\hat{\boldsymbol{x}}(t)$ to be sufficiently accurate and fast for the changes of the real state $\boldsymbol{x}(t)$, the observer dynamics described by (4.24) and expressed by the characteristic eigenvalues of the matrix $\boldsymbol{A}_l$ must be faster than the dynamics of the observed system (4.1), expressed by the characteristic eigenvalues of the matrix $\boldsymbol{A}$. In the case of state space control the dynamics of the observer must be faster than the dynamics of the closed-loop control system.

The observer characteristic polynomial is

$$\begin{aligned} N_{lw}(s) &= \det(s\boldsymbol{I} - \boldsymbol{A}_l) = \\ &= s^n + a_{n-1}^l s^{n-1} + \ldots + a_1^l s + a_0^l = (s - p_1)(s - p_2)\ldots(s - p_n), \end{aligned} \tag{4.31}$$

$$\boldsymbol{a}^l = [a_0^l, a_1^l, \ldots a_{n-1}^l]^T , \tag{4.32}$$

where p_i are the characteristic eigenvalues of the matrix $\boldsymbol{A}_l$ (the observer poles), $\boldsymbol{a}^l$ – the vector of the observer characteristic polynomial coefficients.

Similarly, the characteristic polynomial of the observed system (4.1) is given by (4.2) and the vector $\boldsymbol{a}$ is given by its coefficients (4.13b).

The observer asymptotic stability demands fulfilment of the conditions

$$\operatorname{Re} p_i < 0 \quad \text{for} \quad i = 1, 2, \ldots, n \tag{4.33}$$

and furthermore, in order for the observer to have faster dynamics than the observed system, its all poles p_i must lie to the left of all poles s_i of the observed system, i.e.

$$\min_{1 \le i \le n} \left| \operatorname{Re} p_i \right| > \max_{1 \le i \le n} \left| \operatorname{Re} s_i \right| . \tag{4.34}$$

The convergence $\hat{\boldsymbol{x}}(t) \to \boldsymbol{x}(t)$ will be faster, if there will be greater margin in the inequality (4.34). It is often stated as a decuple, but too great a margin in the inequality (4.34) leads to large values of the components l_i of the state correction vector $\boldsymbol{l}$, and therefore to a large amplification of noise. Therefore, this margin shall be chosen from two-fold to five-fold (it does not apply for integrating systems).

The observer poles are usually chosen as multiple real

$$p_i = -p, \quad p > 0 , \tag{4.35}$$

and therefore the conditions (4.34) can be written in the form

$$p > \max_{1 \le i \le n} \left| \operatorname{Re} s_i \right| . \tag{4.36}$$

In this case, the observer characteristic polynomial in accordance with the binomial theorem has the form

$$N_{lw}(s) = (s + p)^n = \sum_{j=0}^{n} \binom{n}{j} p^j s^{n-j} = s^n + nps^{n-1} + \ldots + np^{n-1}s + p^n . \tag{4.37}$$

Using the observer multiple real pole it ensures the convergence (4.30) with the relative damping equal 1. If it is possible to have very suitable multiple pairs, the selection of multiple pairs

$$-(1 \pm j)p \tag{4.38}$$

will guarantee that the convergence (4.30) will be ensured with the relative damping equal $1/\sqrt{2} \doteq 0.707$. This choice ensures fast convergence and also reduces the value of p. The partial characteristic polynomial

$$s^2 + 2ps + 2p^2 \,. \tag{4.39}$$

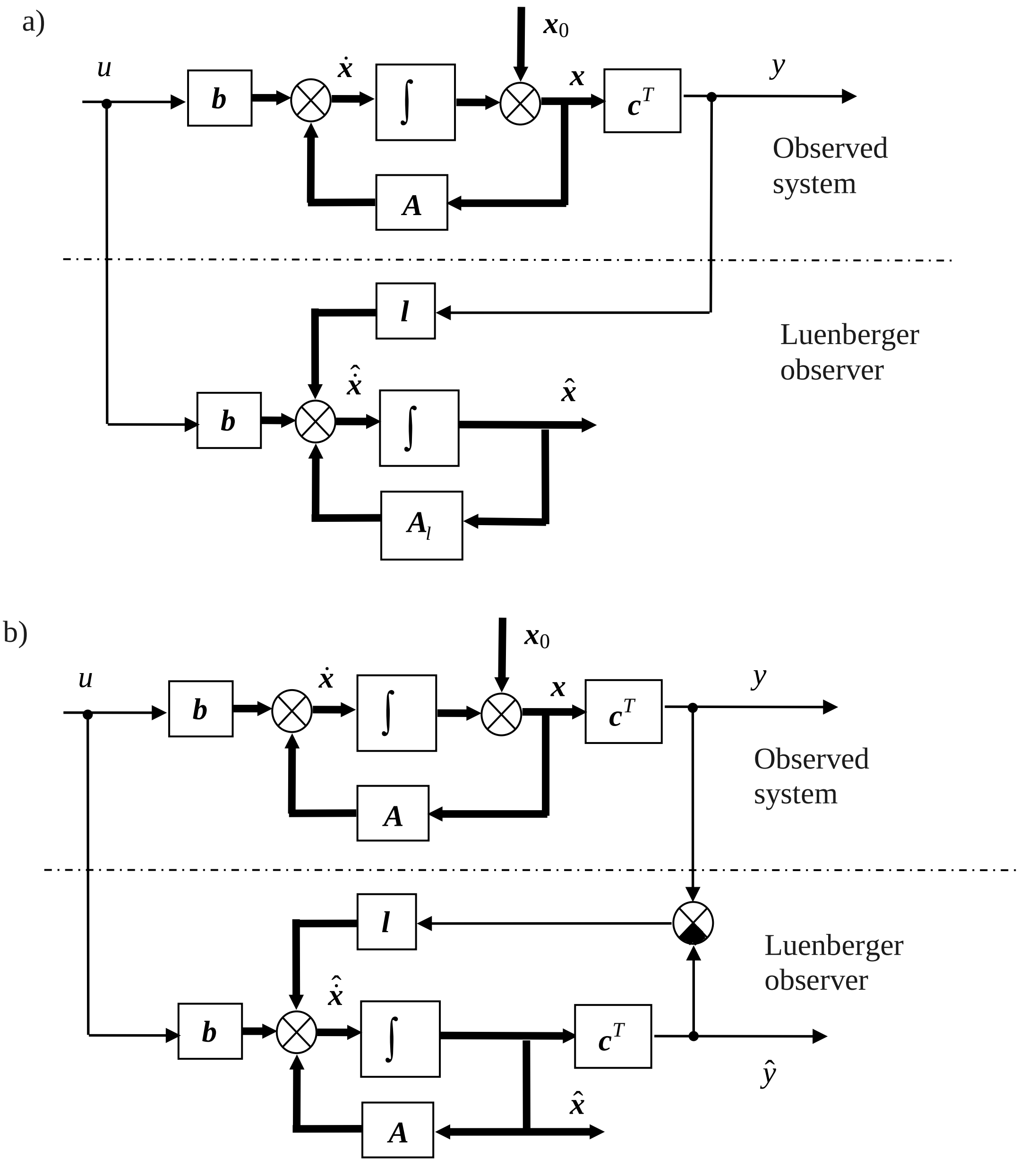


Fig. 4.6 Block diagram of the Luenberger observer: a) original, b) transformed

corresponds to the pair (4.38).

The block diagram in Fig. 4.6a can be transformed in the equivalent block diagram in Fig 4.6b, from which follows the operation of the observer. On the basis of the difference of the output variables $y(t) - \hat{y}(t)$ the state estimate $\hat{\boldsymbol{x}}(t)$ is corrected. It is clear that the Luenberger observer is in fact the model of the observed system with the running feedback correction

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{b}u(t) + \boldsymbol{l}[y(t) - \hat{y}(t)]. \tag{4.40}$$

It is in principle a control system which tries to nullify the difference $y(t) - \hat{y}(t)$, and thus the state error vector $\boldsymbol{\varepsilon}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. Fig. 4.7 shows it clearly. The vector $\boldsymbol{l}$ is therefore also called the Luenberger observer gain vector.

When designing the observer in accordance with the relations (4.24) and (4.27) it is necessary to determine the unknown correction vector (Luenberger observer gain) $\boldsymbol{l}$. It can be determined by comparing the coefficients of the observer characteristic polynomial $N_l(s) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^T)]$ with the corresponding coefficients of the desired observer characteristic polynomial $N_{lw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_l)$ at the same powers of the complex variable *s*. In such a way the system of *n* linear equations is obtained for *n* unknown components l_i of the vector $\boldsymbol{l}$. For large *n*, this procedure is demanding.

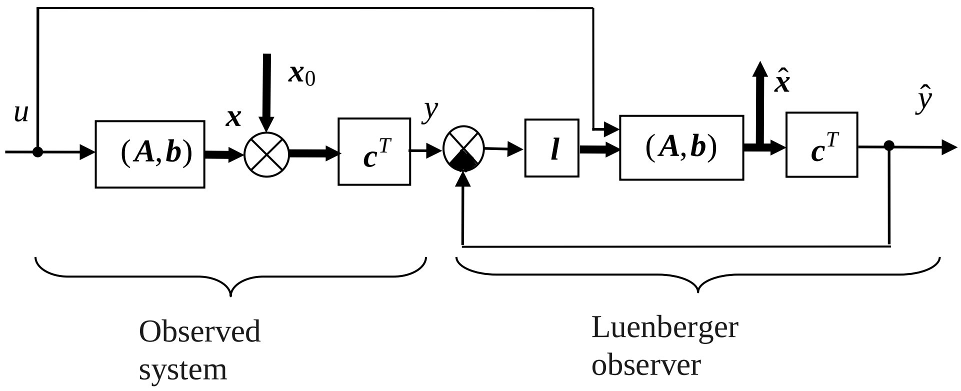


Fig. 4.7 Interpretation of the Luenberger observer

The design of the observer can be easily solved if the model of the observed system (4.1) has the canonical observer form (3.25)

$$\begin{aligned} \dot{\boldsymbol{x}}_o(t) &= \boldsymbol{A}_o\boldsymbol{x}_o(t) + \boldsymbol{b}_o u(t), \\ y(t) &= \boldsymbol{c}_o^T\boldsymbol{x}_o(t), \end{aligned} \tag{4.41a}$$

where

$$\boldsymbol{A}_o = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \tag{4.41b}$$

$$\boldsymbol{b_o} = [b_0, b_1, \dots, b_{n-2}, b_{n-1}]^T, \tag{4.41c}$$

$$\boldsymbol{c}_o^T = [0, 0, \dots, 0, 1]. \tag{4.41d}$$

The canonical observer form can be obtained directly from knowledge of the transfer function (3.22) or using the transformation (3.24)

$$\boldsymbol{x}_o(t) = \boldsymbol{T}_o^{-1}\boldsymbol{x}(t), \quad \boldsymbol{A}_o = \boldsymbol{T}_o^{-1}\boldsymbol{A}\boldsymbol{T}_o\,, \quad \boldsymbol{b}_o = \boldsymbol{T}_o^{-1}\boldsymbol{b}, \quad \boldsymbol{c}_o^T = \boldsymbol{c}^T\boldsymbol{T}_o\,, \tag{4.42}$$

where the regular transformation matrix of the order n [$(n \times n)$]

$$\boldsymbol{T}_o^{-1} = \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) \tag{4.43}$$

is given by the observability matrix of the observed system (4.1), i.e. (2.51) and the matrix $\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)$ is given by the relation (3.24b).

It is clear that by reason of the duality (3.28) it holds

$$\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T) = \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c, \boldsymbol{b}_c)\,. \tag{4.44}$$

The observer (4.24) for (4.27) can also be expressed in the canonical observer form

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_o(t) &= \boldsymbol{A}_{lo}\hat{\boldsymbol{x}}_o(t) + \boldsymbol{b}_o u + \boldsymbol{l}_o y(t), \\ \hat{y}(t) &= \boldsymbol{c}_o^T \hat{\boldsymbol{x}}_o(t), \end{aligned} \tag{4.45a}$$

where

$$\boldsymbol{A}_{lo} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0^l \\ 1 & 0 & \dots & 0 & -a_1^l \\ 0 & 1 & \dots & 0 & -a_2^l \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -a_{n-2}^l \\ 0 & 0 & \dots & 1 & -a_{n-1}^l \end{bmatrix} \tag{4.45b}$$

is the square observer matrix of the order n, in which the negative coefficients of the observer characteristic polynomial (4.31) appear in the last column.

The block diagrams for the canonical observer forms are the same as in Fig. 4.6, but all vectors and matrices must be provided with subscript "o".

In accordance with the relation (4.28) we can write

$$\boldsymbol{A}_{lo} = \boldsymbol{A}_o - \boldsymbol{l}_o\boldsymbol{c}_o^T = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 - l_{o1} \\ 1 & 0 & \dots & 0 & -a_1 - l_{o2} \\ 0 & 1 & \dots & 0 & -a_2 - l_{o3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -a_{n-2} - l_{o,n-1} \\ 0 & 0 & \dots & 1 & -a_{n-1} - l_{on} \end{bmatrix}. \tag{4.46}$$

From a comparison of the relations (4.45b) and (4.46) it follows

$$l_{oi} = a_{i-1}^l - a_{i-1} \quad \text{pro} \quad i = 1,2,\dots,n\,,$$

i.e. in accordance with (4.32) and (4.13b)

$$\boldsymbol{l}_o = \boldsymbol{a}^l - \boldsymbol{a}\,, \tag{4.47}$$

where $\boldsymbol{l}_o$ is the observer correction vector in the canonical observer form.

Therefore (4.42) holds, it is possible to write

$$\boldsymbol{l}_o y = \boldsymbol{T}_o^{-1}\boldsymbol{l}y \;\Rightarrow$$

$$\boldsymbol{l} = \boldsymbol{T}_o \boldsymbol{l}_o = \boldsymbol{T}_o(\boldsymbol{a}^l - \boldsymbol{a})\,. \tag{4.48}$$

From the comparison of the characteristic polynomial of the closed-loop system (4.4) [see also (4.6)]

$$N_{kw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T)]$$

with the characteristic polynomial of the Luenberger observer (4.31) [see also (4.28)]

$$N_l(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_l) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^T)] = \det[s\boldsymbol{I} - (\boldsymbol{A}^T - \boldsymbol{c}\boldsymbol{l}^T)]$$

it follows that for a determination of the Luenberger observer gain vector $\boldsymbol{l}$ the Ackermann's formula [see also (4.16)] can also be used

$$\begin{aligned}\boldsymbol{l}^T &= [0,0,\ldots,0,1][\boldsymbol{c}, \boldsymbol{A}^T\boldsymbol{c}, \ldots, (\boldsymbol{A}^T)^{n-1}\boldsymbol{c}]^{-1} N_{lw}(\boldsymbol{A}) = \\ &= [0,0,\ldots,0,1][\boldsymbol{c}, \boldsymbol{A}^T\boldsymbol{c}, \ldots, (\boldsymbol{A}^T)^{n-1}\boldsymbol{c}]^{-1})[\boldsymbol{A}^n + a_{n-1}^l \boldsymbol{A}^{n-1} + \ldots + a_1^l \boldsymbol{A} + a_0^l \boldsymbol{I}].\end{aligned}$$

or

$$\begin{aligned}\boldsymbol{l} &= N_{lw}(\boldsymbol{A})\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T)\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} = \\ &= [\boldsymbol{A}^n + a_{n-1}^l \boldsymbol{A}^{n-1} + \ldots + a_1^l \boldsymbol{A} + a_0^l \boldsymbol{I}]\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T)\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}.\end{aligned} \tag{4.49}$$

Consider now, that the state space controller uses the state estimate $\hat{\boldsymbol{x}}(t)$ for control (Fig. 4.8), i.e.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{b}\boldsymbol{k}^T\hat{\boldsymbol{x}}(t)\,.$$

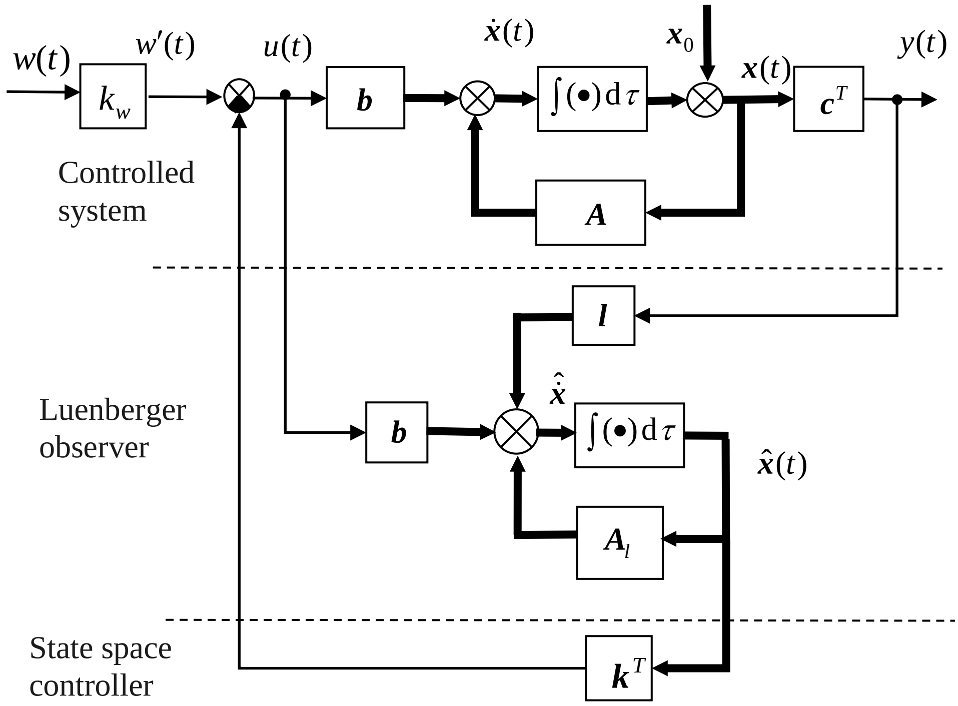


Fig. 4.8 Block diagram of a control system with a state space controller and Luenberger state observer

Therefore the equality holds

$$-\boldsymbol{bk}^T\hat{\boldsymbol{x}}(t) = -\boldsymbol{bk}^T\boldsymbol{x}(t) + \boldsymbol{bk}^T\boldsymbol{\varepsilon}(t)\,,$$

we can write the state equation of the control system with state space controller and the Luenberger observer in the form [see (4.6)]

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_w\boldsymbol{x}(t) + \boldsymbol{bk}^T\boldsymbol{\varepsilon}(t),\\ \dot{\boldsymbol{\varepsilon}}(t) &= \boldsymbol{A}_l\boldsymbol{\varepsilon}(t),\end{aligned} \tag{4.50a}$$

or

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{\varepsilon}}(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_w & \boldsymbol{bk}^T\\ \boldsymbol{0} & \boldsymbol{A}_l\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{\varepsilon}(t)\end{bmatrix}. \tag{4.50b}$$

It is the upper block triangular matrix, whose characteristic polynomial is given by the relation

$$N_k(s)N_l(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w)\det(s\boldsymbol{I} - \boldsymbol{A}_l)\,. \tag{4.51}$$

This means that the dynamic properties of the control system with the state space controller and the Luenberger state observers are mutually independent.

It is the so called **separation principle**.

It is very important because the state observer and the state space controller can be designed independently. We can design a state space controller that ensures the required control performance and then we can separately design the Luenberger state observer, which ensures the correct state variable estimates. A well-designed state observer deteriorates the resulting dynamics of a control system with a state space controller very little.

Procedure:

1. Check the controllability and observability of the controlled system (plant) [relations (2.50) and (2.51)].
2. Determine the coefficients of the characteristic polynomials $N(s)$ and $N_{lw}(s)$ [relations (4.2) and (4.31)].
3. On the basis of the pole of the control system with the largest absolute real part determine the multiple pole (4.36) or multiple pairs of poles (4.38) in such a way to ensure the sufficiently fast dynamics of the observer.
4. Compare the coefficients of the observer characteristic polynomial $N_l(s) = \det[s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{lc}^T)]$ with the corresponding coefficients of the desired observer characteristic polynomial $N_{lw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_l)$ at the same powers of the complex variable s and the solution of the system of n linear equations is obtained for n unknown components l_i of the vector $\boldsymbol{l}$. For large n, use the transformation matrix (4.43) and the relation (4.48) or Ackermann's formula (4.49).
5. Verify by simulating the received estimates of the state variables

Example 4.3

For the control system with the state space controller from Example 4.1 it is necessary to design the Luenberger state observer.

Solution:

The poles of the controlled linear dynamic system (4.17) are $s_{1,2} = \pm\sqrt{2}$ therefore, in accordance with the conditions of (4.34) − (4.36), we choose, e.g.

$$p = 4 \;\Rightarrow\; p_1 = p_2 = -4 .$$

The characteristic polynomial of the observer will be

$$N_{lw}(s) = (s+p)^2 = s^2 + 8s + 16 \;\Rightarrow$$
$$a_0^l = 16,\; a_1^l = 8 \;\Rightarrow\; \boldsymbol{a}^l = [16 \quad 8]^T .$$

Method of comparison of coefficient

The system (dynamics) matrix of the observer is given by the relation (4.28)

$$\boldsymbol{A}_l = \boldsymbol{A} - \boldsymbol{lc}^T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1-l_1 & 1 \\ 1-l_2 & 1 \end{bmatrix}.$$

Now we can calculate the characteristic polynomial of the observer (4.31)

$$N_l(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_l) = \begin{bmatrix} s+1+l_1 & -1 \\ -1+l_2 & s-1 \end{bmatrix} = s^2 + l_1 s - 2 - l_1 + l_2 .$$

We compare coefficients of both characteristic polynomials $N_l(s)$ and $N_{lw}(s)$, i.e.

$$\left.\begin{array}{r} l_1 = 8 \\ -2 - l_1 + l_2 = 16 \;\Rightarrow\; l_2 = 26 \end{array}\right\} \Rightarrow \boldsymbol{l} = \begin{bmatrix} 8 \\ 26 \end{bmatrix}.$$

Method of transformation

We will use the relation (4.48) for $\boldsymbol{a}^l = [16 \quad 8]^T$ and $\boldsymbol{a} = [-2 \quad 0]^T$:

$$\boldsymbol{T}_o^{-1} = \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_o = \frac{\operatorname{adj}\boldsymbol{T}_o^{-1}}{\det \boldsymbol{T}_o^{-1}} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\boldsymbol{l} = \boldsymbol{T}_o(\boldsymbol{a}^l - \boldsymbol{a}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\left\{\begin{bmatrix} 16 \\ 8 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right\} = \begin{bmatrix} 8 \\ 26 \end{bmatrix}.$$

We received the same result.

Ackermann's formula

In accordance with (4.49) we can write:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \; \boldsymbol{A}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \; \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T) = \frac{\operatorname{adj}\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T)}{\det \boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T)} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\boldsymbol{l} = [\boldsymbol{A}^2 + a_1^l\boldsymbol{A} + a_0^l\boldsymbol{I}]\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T)\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \left\{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 8\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + 16\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 26 \end{bmatrix}$$

As we expected, we received the same result as in the previous two cases.

The system matrix of the observer for determined Luenberger gain vector $\boldsymbol{l}$ is

$$\boldsymbol{A}_l = \begin{bmatrix} -1 - l_1 & 1 \\ 1 - l_2 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -25 & 1 \end{bmatrix}.$$

The block diagram of the control system with state controller and the Luenberger observer is shown in Fig. 4.9

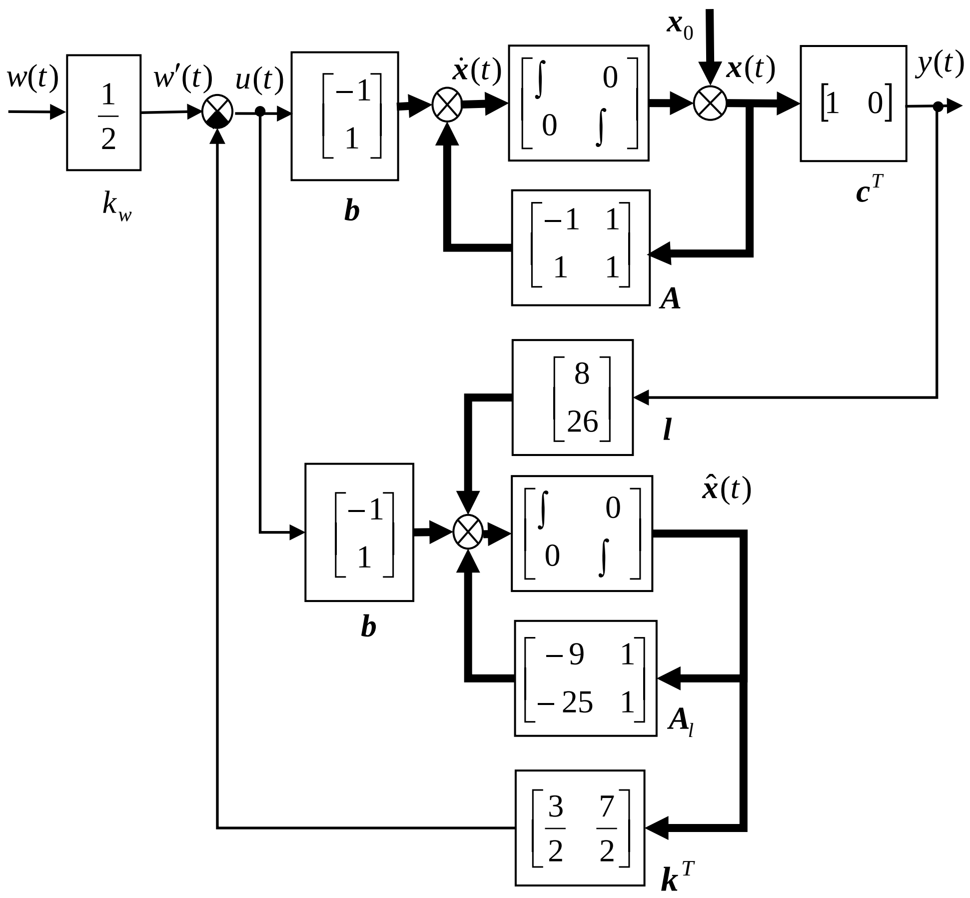


Fig. 4.9 Block diagram of control system with state controller and Luenberger observer – Example 4.3

The step response ($\boldsymbol{x}_0 = \boldsymbol{0}$) of the control system with the state controller and the Luenberger observer is the same as without the observer (see Fig. 4.4).

Example 4.4

For the control system with state space controller from Example 4.2 it is necessary to design the Luenberger state observer.

Solution:

In the Example 4.2 it was shown that the controlled system is controllable and observable, and that its characteristic polynomial has the form

$$N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = s^3 + 7s^2 + 14s + 8 = (s+1)(s+2)(s+4),$$

where

$$s_1 = -1,\ s_2 = -2,\ s_3 = -4$$

are the controlled system poles and

$$a_0 = 8,\ a_1 = 14,\ a_2 = 7 \quad \Rightarrow \quad \boldsymbol{a} = [8,\ 14,\ 7]^T$$

are its characteristic polynomial coefficients or the vector of these coefficients.

Since

$$\max_{1\le i\le 3}|s_i| = 4$$

it is possible to choose

$$p_1 = p_2 = p_3 = -p = -8$$

i.e. the observer characteristic polynomial and its coefficients are

$$N_{lw}(s) = (s+p)^3 = (s+8)^3 = s^3 + 24s^2 + 192s + 512 \Rightarrow$$

$$a_0^l = 512,\ a_1^l = 192,\ a_2^l = 24 \Rightarrow \boldsymbol{a}^l = [512,\quad 192,\quad 24]^T .$$

Method of comparison of coefficient

The observer system (dynamics) matrix is

$$\boldsymbol{A}_l = \boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}^T = \begin{bmatrix} -1 & 0 & -4 \\ 2 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} -2 & 4 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2l_1 - 1 & -4l_1 & -l_1 - 4 \\ 2l_2 + 2 & -4l_2 - 2 & -l_2 - 2 \\ 2l_3 & -4l_3 & -l_3 - 4 \end{bmatrix}.$$

After the unpleasant and time-consuming modifications we can determine the characteristic polynomial of the observer

$$N_l(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_l) =$$

$$= s^3 + (-2l_1 + 4l_2 + l_3 + 7)s^2 + (-4l_1 + 20l_2 + 3l_3 + 14)s + 16l_1 + 16l_2 - 22l_3 + 8.$$

Comparing the coefficients at the same powers of the complex variable s for both of the observer characteristic polynomials, the system of linear algebraic equations with respect to unknown components l_1, l_2 and l_3 of the observer correction vector $\boldsymbol{l}$ was obtained, i.e.

$$\left.\begin{aligned} 16l_1 + 16l_2 - 22l_3 &= 504 \\ -4l_1 + 20l_2 + 3l_3 &= 178 \\ -2l_1 + 4l_2 + l_3 &= 17 \end{aligned}\right\} \Rightarrow \begin{aligned} l_1 &= \frac{773}{54}, \\ l_2 &= \frac{332}{27}, \\ l_3 &= -\frac{32}{9}. \end{aligned} \Rightarrow \boldsymbol{l} = \begin{bmatrix} \frac{773}{54} \\ \frac{332}{27} \\ -\frac{32}{9} \end{bmatrix}.$$

Method of transformation

In accordance with (4.43) and (4.44) there is obtained

$$\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T) = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The transformation matrix can now be determined

$$\boldsymbol{T}_o^{-1} = \boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}_o, \boldsymbol{c}_o^T)\boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} 16 & 16 & -22 \\ -4 & 20 & 3 \\ -2 & 4 & 1 \end{bmatrix} \Rightarrow$$

$$\boldsymbol{T}_o = \begin{bmatrix} -\frac{1}{54} & \frac{13}{54} & -\frac{61}{54} \\ \frac{1}{216} & \frac{7}{108} & -\frac{5}{54} \\ -\frac{1}{18} & \frac{2}{9} & -\frac{8}{9} \end{bmatrix}.$$

After substituting into the relation on the observer correction (gain) vector $\boldsymbol{l}$, the same result

$$\boldsymbol{l} = \boldsymbol{T}_o(\boldsymbol{a}^l - \boldsymbol{a}) = \begin{bmatrix} \frac{773}{54} \\ \frac{332}{27} \\ -\frac{32}{9} \end{bmatrix}.$$

is obtained.

Ackermann’s formula

We use the Ackermann‘s formula (4.49) and the partial results from Example 4.2:

$$N_{lw}(\boldsymbol{A}) = [\boldsymbol{A}^3 + 24\boldsymbol{A}^2 + 192\boldsymbol{A} + 512\boldsymbol{I}] = \begin{bmatrix} 343 & 0 & -372 \\ 254 & 216 & -288 \\ 0 & 0 & 64 \end{bmatrix},$$

$$\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T) = \begin{bmatrix} -2 & 4 & 1 \\ 10 & -8 & -4 \\ -26 & 16 & -8 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{8}{27} & \frac{1}{9} & -\frac{1}{54} \\ \frac{23}{54} & \frac{7}{72} & \frac{1}{216} \\ -\frac{1}{9} & -\frac{1}{6} & -\frac{1}{18} \end{bmatrix},$$

$$\boldsymbol{l} = N_{lw}(\boldsymbol{A})\boldsymbol{Q}_{ob}^{-1}(\boldsymbol{A}, \boldsymbol{c}^T)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{773}{54} \\ \frac{332}{27} \\ -\frac{32}{9} \end{bmatrix}.$$

We see that in all three cases we obtained the same results.

The step response of the control system with the state space controller and the Luenberger state observer and without the Luenberger state observer is shown in Fig. 4.10, from which it is clear that the designed observer operates correctly.

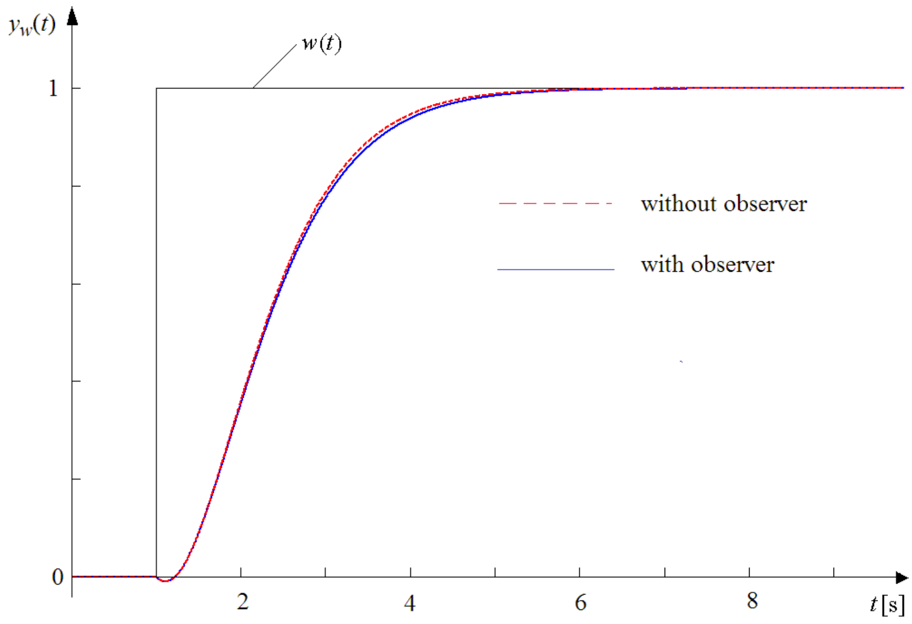


Fig. 4.10 Influence of the Luenberger state observer on the step response of a control system with a state space controller – Example 4.4

4.3 Integral state space control

A state space controller is able to ensure the required pole placement of a control system, this means that it is able to ensure its dynamic properties but it can not eliminate a harmful effect of disturbance variables.

If disturbances $\boldsymbol{v}(t)$ exist, the state model of the controlled system has the following form

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) + \boldsymbol{F}\boldsymbol{v}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ &y(t) = \boldsymbol{c}^T \boldsymbol{x}(t), \end{aligned} \tag{4.52}$$

where $\boldsymbol{v}(t)$ is the disturbance vector of dimension p, $\boldsymbol{F}$ – the disturbance matrix of dimension $(n \times p)$.

In order to remove the disturbance $\boldsymbol{v}(t)$, we add another loop with the I or PI controller, see Fig. 4.11, where the K_I is the weight of the integral component. It is obvious that the number of poles is increased by 1.

In accordance with Fig. 4.11 and relations (4.52) the control system with the integral state space controller can be described by the state model (the time t we will not expres)

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T & \boldsymbol{b}K_I \\ -\boldsymbol{c}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix} w + \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{0}^T \end{bmatrix} \boldsymbol{v} \tag{4.53a}$$

$$y = \begin{bmatrix} \boldsymbol{c}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ x_{n+1} \end{bmatrix}. \tag{4.53b}$$

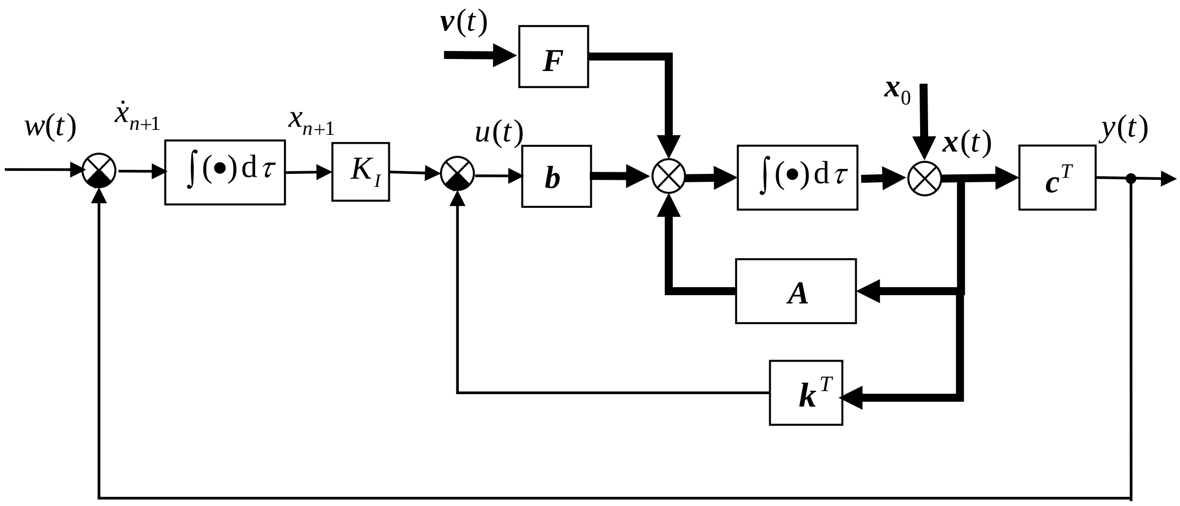


Fig. 4.11 Block diagram of control system with state space controller and added loop with I controller for removing disturbances

In order to exploit the results of the previous sections 4.1 and 4.2 we rewrite the system matrix (4.53)

$$\begin{bmatrix} \boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^T & \boldsymbol{b}K_I \\ -\boldsymbol{c}^T & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{c}^T & 0 \end{bmatrix}}_{\boldsymbol{A}_e} - \underbrace{\begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix}}_{\boldsymbol{b}_e} \underbrace{\begin{bmatrix} \boldsymbol{k}^T & -K_I \end{bmatrix}}_{\boldsymbol{k}_e^T} \tag{4.54}$$

and we get the extended state model of the controlled system

$$\begin{aligned} \dot{\boldsymbol{x}}_e &= \boldsymbol{A}_e\boldsymbol{x}_e + \boldsymbol{b}_e u + \boldsymbol{F}\boldsymbol{v}, \\ y &= \boldsymbol{c}_e^T\boldsymbol{x}_e, \end{aligned} \tag{4.55a}$$

where

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{x} \\ x_{n+1} \end{bmatrix},\ \boldsymbol{A}_e = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{c}^T & 0 \end{bmatrix},\ \boldsymbol{b}_e = \begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix},\ \boldsymbol{c}_e^T = \begin{bmatrix} \boldsymbol{c}^T & 0 \end{bmatrix} \tag{4.55b}$$

The extended controlled system has the property that, when we use

$$u = \boldsymbol{k}_e^T\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{k}^T & -K_I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ x_{n+1} \end{bmatrix}$$

we get the equation (4.53a)

The characteristic polynomial of the extended controlled system (4.55) is given by relation

$$\begin{aligned} N_e(s) &= \det(s\boldsymbol{I}-\boldsymbol{A}_e) = s(s-s_1)(s-s_2)\ldots(s-s_n) = \\ &= s^{n+1} + a_{en}s^n + \ldots + a_{e1}s \ \Rightarrow\ \boldsymbol{a}_e = [0, a_{e1}, a_{e2}, \ldots, a_{en}]^T \end{aligned} \tag{4.56}$$

and the desired characteristic polynomial of the closed-loop control system is

$$N_{ew}(s)=\det[s\boldsymbol{I}-(\boldsymbol{A}_e-\boldsymbol{b}_e\boldsymbol{k}_e^T)]=(s-s_1^w)(s-s_2^w)\dots(s-s_{n+1}^w)=$$
$$=s^{n+1}+a_{en}^w s^n+\dots+a_{e1}^w s+a_{e0}^w \Rightarrow \boldsymbol{a}_e^w=[a_{e0}^w,a_{e1}^w,\dots,a_{en}^w]^T, \tag{4.57}$$

where

$$\boldsymbol{k}_e^T=\begin{bmatrix}\boldsymbol{k}^T & -K_I\end{bmatrix} \tag{4.58}$$

is the vector of the feedback state space controller and s_i^w are the required poles of the closed-loop control system (i = 1, 2,…, n + 1).

Procedure:

1. Check the controllability and observability of the controlled system (plant) [relations (2.50) and (2.51)].
2. Modify the original state model of the controlled system (4.52) to the extended state model (4.55).
3. Formulate the requirements for the control performance and express it by the desired pole placement (i.e. by the characteristic polynomial) of the closed-loop control system for the extended state model (4.55).
4. Determine the coefficients of the characteristic polynomials $N_e(s)$ and $N_{ew}(s)$ [relations (4.56) and (4.57)].
5. Determine the extended feedback vector $\boldsymbol{k}_e^T$ (4.58) by any method from the section 4.1.
6. Verify the received control performance by a simulation.

Example 4.5

For DC motor from Example 3.2

$$\begin{bmatrix}\dfrac{\mathrm{d}\alpha(t)}{\mathrm{d}t}\\ \dfrac{\mathrm{d}\omega(t)}{\mathrm{d}t}\\ \dfrac{\mathrm{d}i_a(t)}{\mathrm{d}t}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -\dfrac{b_m}{J_m} & \dfrac{c_m}{J_m}\\ 0 & -\dfrac{c_e}{L_a} & -\dfrac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}\alpha(t)\\ \omega(t)\\ i_a(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \dfrac{1}{L_a}\end{bmatrix}u_a(t)-\begin{bmatrix}0\\ \dfrac{1}{J_m}\\ 0\end{bmatrix}m_l(t)$$

it is necessary to design a state space control without and with an integration for the angle of the motor shaft $\alpha(t)$ for the parameters: J_m = 0.02 kg m^2, b_m = 0.01 N m s rad^{-1}, $c_m = c_e$ = 0.05 N m A^{-1} (V s rad^{-1}), L_a =0.2 H, R_a = 1 Ω. For the step change of the desired angle $\alpha_w(t)$ the course of $\alpha(t)$ is required without overshoot.

Solution:

Because the angle $\alpha(t)$, the angular velocity $\omega(t)$ and the armature current $i_a(t)$ are relatively well directly measurable quantities, the observer will not be proposed.

For greater clarity we use standard symbols

$$x_1=\alpha,\ \ x_2=\omega,\ \ x_3=i_a,\ \ u=u_a,\ \ v=m_l$$

and by substitution of the numerical values of the DC motor parameters we get its state model in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{f}v,$$
$$y = \boldsymbol{c}^T \boldsymbol{x},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 2{,}5 \\ 0 & -0.25 & -5 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \ \boldsymbol{f} = \begin{bmatrix} 0 \\ -50 \\ 0 \end{bmatrix}, \ \boldsymbol{c}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

We verify the controllability and the observability:

$$\boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}] = \begin{bmatrix} 0 & 0 & 12.5 \\ 0 & 12.5 & -68.75 \\ 5 & -25 & 121.875 \end{bmatrix}, \det \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b}) = -781.25 \Rightarrow$$

The DC motor is controllable.

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \boldsymbol{c}^T\boldsymbol{A}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 2.5 \end{bmatrix}, \ \det \boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T) = 2.5 \Rightarrow$$

The DC motor is observable.

We determine the characteristic polynomial of the DC motor

$$N(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s+0.5 & -2.5 \\ 0 & 0.25 & s+5 \end{vmatrix} = s^3 + 5.5s^2 + 3.125s \ \Rightarrow$$
$$s_1 = 0, \ s_2 \doteq -0.6435, \ s_2 \doteq -4.8565,$$
$$a_0 = 0, \ a_1 = 3.125, \ a_2 = 5.5 \ \Rightarrow \ \boldsymbol{a} = [0 \quad 3.125 \quad 5.5]^T.$$

State space control without integration

Due to the requirements for the course of angle $\alpha(t)$ without overshoot, we choose the multiple pole of the closed-loop control system $s^w_{1,2,3} = -5$ and we get the desired characteristic polynomial

$$N_{kw}(s) = (s+5)^3 = s^3 + 15s^2 + 75s + 125 \Rightarrow$$
$$a^w_0 = 125, \ a^w_1 = 75, \ a^w_2 = 15 \ \Rightarrow \ \boldsymbol{a}^w = [125 \quad 75 \quad 15]^T.$$

For the design of the state space control without the integration we use e.g. the relation (4.14):

$$\boldsymbol{Q}^{-1}_{co}(\boldsymbol{A}_c,\boldsymbol{b}_c) = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3.125 & 5.5 & 1 \\ 5.5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}_c,\boldsymbol{b}_c) = \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 12.5 & 0 \\ 0 & 2.5 & 5 \end{bmatrix},$$

$$\boldsymbol{T}_c^{-1} = \begin{bmatrix} 0.08 & 0 & 0 \\ 0 & 0.08 & 0 \\ 0 & -0.04 & 0.2 \end{bmatrix}.$$

$$\left(\boldsymbol{a}^w - \boldsymbol{a}\right)^T = \begin{bmatrix} 125 & 71.875 & 9.5 \end{bmatrix},$$

$$\boldsymbol{k}^T = \left(\boldsymbol{a}^w - \boldsymbol{a}\right)^T \boldsymbol{T}_c^{-1} = \begin{bmatrix} 10 & 5.37 & 1.9 \end{bmatrix}.$$

We determine the system matrix of the closed-loop control system

$$\boldsymbol{A}_w = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 2.5 \\ -50 & -27.1 & -14.5 \end{bmatrix}.$$

We can easily verify that eigenvalues of the matrix A_w are $s_{1,2,3}^w = -5$, i.e. they are realy desired poles of the closed-loop control system.

We determine the input correction on the basis of the relation (4.9)

$$k_w = -\frac{1}{\boldsymbol{c}^T \boldsymbol{A}_w^{-1} \boldsymbol{b}} = 10\,.$$

The block diagram of the state space control without the integration of the DC motor is shown in Fig. 4.12.

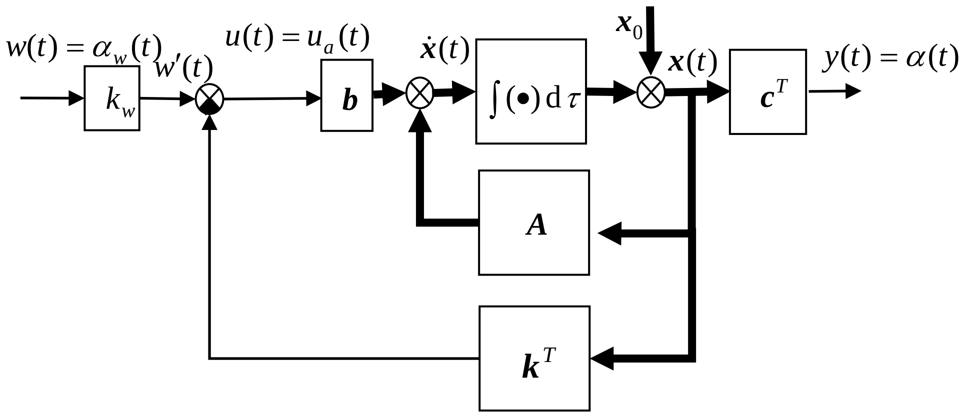


Fig. 4.12 Block diagram of state space control without integration of DC motor – Example 4.5

The DC motor responses with the state space control without the integration for the step change of the desired angle $w(t) = \alpha_w(t) = \eta(t)$ and the step change of the load torque $v(t) = m_l(t) = 0.1\eta(t-5)$ is shown in Fig. 4.13.

State space control with integration

In this example for simplicity we also select the multiple pole of the closed-loop control system $s^w_{1,2,3} = -5$, i.e.

$$N_{ew}(s) = (s+5)^4 = s^4 + 20s^3 + 150s^2 + 500s + 625 \Rightarrow$$
$$a^w_{e0} = 625,\ a^w_{e1} = 500,\ a^w_{e2} = 150,\ a^w_{e3} = 20 \Rightarrow \boldsymbol{a}^w = [625 \quad 500 \quad 150 \quad 20]^T.$$

Now we must consider an extended state model of the DC motor in the form (4.55), i.e.

$$\boldsymbol{A}_e = \begin{bmatrix} \boldsymbol{A} & 0 \\ -\boldsymbol{c}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.5 & 2.5 & 0 \\ 0 & -0.25 & -5 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},\ \boldsymbol{b}_e = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix},$$
$$\boldsymbol{c}^T_e = [1 \quad 0 \quad 0 \quad 0].$$

We determine the characteristic polynomial

$$N_e(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_e) = \begin{vmatrix} s & -1 & 0 & 0 \\ 0 & s+0.5 & -2.5 & 0 \\ 0 & 0.25 & s+5 & 0 \\ 1 & 0 & 0 & s \end{vmatrix} = s^4 + 5.5s^3 + 3.125s^2 \Rightarrow$$
$$a_{e0} = 0,\ a_{e1} = 0,\ a_{e2} = 3.125,\ a_{e3} = 5.5 \Rightarrow \boldsymbol{a}_e = [0 \quad 0 \quad 3.125 \quad 5.5].$$

For the design of the state space controller

$$\boldsymbol{k}^T_e = [\boldsymbol{k}^T \quad -K_I]$$

we also use the relation (4.14):

$$\boldsymbol{Q}_{co}(\boldsymbol{A}_e, \boldsymbol{b}_e) = [\boldsymbol{b}_e, \boldsymbol{A}_e\boldsymbol{b}_e, \boldsymbol{A}^2_e\boldsymbol{b}_e, \boldsymbol{A}^3_e\boldsymbol{b}_e] = \begin{bmatrix} 0 & 0 & 12.5 & -68.75 \\ 0 & 12.5 & -68.75 & 339.0625 \\ 5 & -25 & 121.875 & -592.1875 \\ 0 & 0 & 0 & -12.5 \end{bmatrix},$$

$$\boldsymbol{Q}^{-1}_{co}(\boldsymbol{A}_{ec}, \boldsymbol{b}_{ec}) = \begin{bmatrix} a_{e1} & a_{e2} & a_{e3} & 1 \\ a_{e2} & a_{e3} & 1 & 0 \\ a_{e3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3.125 & 5{,}5 & 1 \\ 3.125 & 5.5 & 1 & 0 \\ 5.5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_c = \boldsymbol{Q}_{co}(\boldsymbol{A}_e, \boldsymbol{b}_e)\boldsymbol{Q}^{-1}_{co}(\boldsymbol{A}_{ec}, \boldsymbol{b}_{ec}) = \begin{bmatrix} 0 & 12.5 & 0 & 0 \\ 0 & 0 & 1.25 & 0 \\ 0 & 0 & 2.5 & 5 \\ -12.5 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{T}_c^{-1} = \begin{bmatrix} 0 & 0 & 0 & -0.08 \\ 0.08 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & -0.4 & 0.2 & 0 \end{bmatrix},$$

$$\left(\boldsymbol{a}_e^w - \boldsymbol{a}_e\right)^T = \begin{bmatrix} 625 & 500 & 146.875 & 14.5 \end{bmatrix},$$

$$\boldsymbol{k}_e^T = \left(\boldsymbol{a}_e^w - \boldsymbol{a}_e\right)^T \boldsymbol{T}_c^{-1} = \begin{bmatrix} 40 & 11.17 & 2.9 & -50 \end{bmatrix}.$$

The block diagram of the state space control with the integration of the DC motor is the same as in Fig. 4.11.

The block diagram of the state space control with the integration of the DC motor is shown in Fig. 4.11.

The DC motor responses with the state space control with the integration for the step change of the desired angle $w(t) = \alpha_w(t) = \eta(t)$ and the step change of the load torque $v(t) = m_l(t) = 0.1\eta(t-5)$ is shown in Fig. 4.13.

The comparison of the courses in Fig. 4.13 shows unambiguous priority of the state space control with the integration although there is a response slowing. The response slowing is due to an increase of the closed-loop control system order.

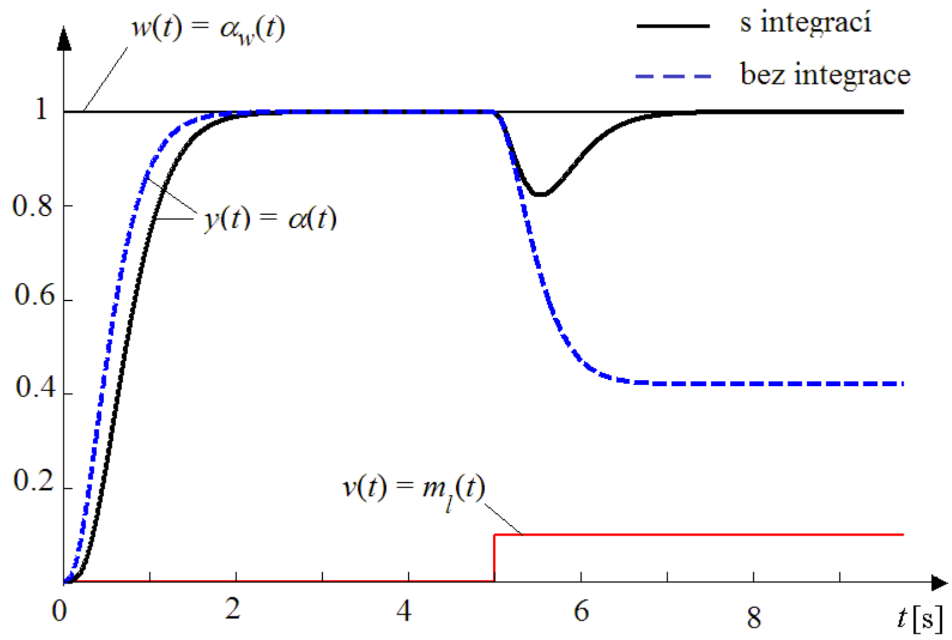


Fig. 4.13 Comparison of responses of DC motor with state space control without and with integration – Example 4.5

APPENDIX A

Linearization

Linear dynamic systems are basically an idealization of real dynamic systems. One of the most important requirements is that the system is working in "close" surroundings of the **operating point**. Then the mathematical model of this dynamic system can be considered as linear in this surroundings.

When the mathematical model of the nonlinear dynamical system is given by relation (2.8) in the state space, i.e.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}[\boldsymbol{x}(t), u(t)],$$

$$y(t) = h[\boldsymbol{x}(t), u(t)],$$

then it is necessary to provide the **linearization**. We use expansion in Taylor series and we consider only the first linear terms of this series, then we can write

$$\left.\begin{aligned} \Delta\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\Delta\boldsymbol{x}(t) + \boldsymbol{b}\Delta u(t), \\ \Delta y(t) &= \boldsymbol{c}^T\Delta\boldsymbol{x}(t) + d\Delta u(t), \end{aligned}\right\} \tag{A.1a}$$

where

$$\left.\begin{aligned} &\Delta\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(t), && \Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_0, \\ &\Delta u(t) = u(t) - u_0, && \\ &\boldsymbol{A} = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_0, && \boldsymbol{b} = \left.\frac{\partial \boldsymbol{g}}{\partial u}\right|_0, \\ &\boldsymbol{c} = \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_0, && d = \left.\frac{\partial h}{\partial u}\right|_0. \end{aligned}\right\} \tag{A.1b}$$

In all cases it is assumed that the partial derivatives are calculated for the operating point, and that they exist and they are continuous.

The transition from incremental values of the variables to the absolute values of the variables is given by the relations

$$\left.\begin{aligned} \hat{y}(t) &= y_0 + \Delta y(t), \\ u(t) &= u_0 + \Delta u(t). \end{aligned}\right\} \tag{A.2}$$

Throughout the text, unless otherwise stated, all mathematical models are considered that they are working in the operating point, i.e. we use incremental variables, although it is not explicitly stated, and the variables are not designated as incremental.

For more details see e.g. [deSilva 2009; Mandal 2006; Víteček, Vítečková 2013].

APPENDIX B

Cayley-Hamilton theorem

Every square matrix $\boldsymbol{A}$ of order n satisfies its own characteristic equation

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \,, \tag{B.1}$$

$$\boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \ldots + a_1\boldsymbol{A} + a_0\boldsymbol{I} = \boldsymbol{0} \,. \tag{B.2}$$

For more details see e.g. [Ogata 2010; Mandal 2006].

Sylvester interpolation formula

The convergent infinite series [see e.g. (3.40)]

$$f(\boldsymbol{A}) = \sum_{i=0}^{\infty} \alpha_i \boldsymbol{A}^i \tag{B.3}$$

of square matrices of order n can be uniquely expressed in the finite series of degree $n - 1$ or less

$$f(\boldsymbol{A}) = \sum_{i=0}^{n-1} \alpha_i \boldsymbol{A}^i \,, \tag{B.4}$$

where the coefficients α_i are functions of the eigenvalues of the matrix $\boldsymbol{A}$. For more details see e.g. [Ogata 2010; Mandal 2006].

APPENDIX C

Consider the linear dynamic system with the state model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0\,, \tag{C.1a}$$

$$y(t) = \boldsymbol{c}^T \boldsymbol{x}(t) + du(t)\,, \tag{C.1b}$$

whose state response is given by the relation (3.43)

$$\boldsymbol{x}(t) = \underbrace{\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}(0)}_{\substack{\text{free state}\\ \text{response}}} + \underbrace{\mathrm{e}^{\boldsymbol{A}t}\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\,\boldsymbol{b}u(\tau)\,\mathrm{d}\,\tau}_{\substack{\text{forced state}\\ \text{response}}} \tag{C.2}$$

and the output response by the relation (3.44)

$$y(t) = \underbrace{\boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}(0)}_{\substack{\text{free output}\\ \text{response}}} + \underbrace{\boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\,\boldsymbol{b}u(\tau)\,\mathrm{d}\,\tau + du(t)}_{\substack{\text{forced output}\\ \text{response}}}. \tag{C.3}$$

Controllability

A linear dynamic system is controllable if there is such a control (input) $u(t)$ which transfers the system from any initial state $\boldsymbol{x}(t_0)$ to any final state $\boldsymbol{x}(t_1)$ in a finite time $t_1 - t_0$.

Most often it is selected $t_0 = 0$ a $\boldsymbol{x}(t_1) = \boldsymbol{0}$.

It is clear that for the controllability the output equation (C.1b) [(C.3)] has not significance and therefore it is considered only the equation (C.1a) and its response (C.2)

In accordance with (C.2) for the final state $\boldsymbol{x}(t_1) = \boldsymbol{0}$ it holds

$$\boldsymbol{0} = \mathrm{e}^{\boldsymbol{A}t_1}\,\boldsymbol{x}(0) + \mathrm{e}^{\boldsymbol{A}t_1}\int_0^{t_1} \mathrm{e}^{-\boldsymbol{A}\tau}\,\boldsymbol{b}u(\tau)\,\mathrm{d}\,\tau \;\Rightarrow$$

$$\boldsymbol{x}(0) = -\int_0^{t_1} \mathrm{e}^{-\boldsymbol{A}\tau}\,\boldsymbol{b}u(\tau)\,\mathrm{d}\,\tau. \tag{C.4}$$

Applying Sylvester interpolation formula (B.4)

$$\mathrm{e}^{-\boldsymbol{A}\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)\boldsymbol{A}^i\,, \tag{C.5}$$

and substituting into (C.4) then we get

$$\boldsymbol{x}(0) = -\int_0^{t_1} \sum_{i=0}^{n-1} \alpha_i(\tau)\boldsymbol{A}^i \boldsymbol{b}u(\tau)\,\mathrm{d}\,\tau \;\Rightarrow$$

$$\boldsymbol{x}(0) = \sum_{i=0}^{n-1} \boldsymbol{A}^i \boldsymbol{b}\beta_i\,, \tag{C.6a}$$

$$\beta_i = -\int_0^{t_1} \alpha_i(\tau) u(\tau)\, \mathrm{d}\tau \,. \qquad \text{(C.6b)}$$

The relation (C.6a) can be written in the form

$$\boldsymbol{x}(0) = \left[\boldsymbol{b}, \boldsymbol{Ab}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b}\right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{x}(0) = \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}, \qquad \text{(C.7)}$$

where

$$\boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = \left[\boldsymbol{b}, \boldsymbol{Ab}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b}\right] \qquad \text{(C.8)}$$

is the square controllability matrix [see relation (2.50)].

From the relation (C. 7) it follows that in order to determine β_0, $\beta_1, \ldots, \beta_{n-1}$, the controllability matrix (C. 8) must be invertible, i.e. it must have the rank n. Since it is a square matrix, its determinant must be zero.

$$\operatorname{rank} \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = n \;\Leftrightarrow\; \det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \neq 0 \,. \qquad \text{(C.9)}$$

This follows directly from the known relation for inversion of a square matrix

$$\boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}, \boldsymbol{b}) = \frac{\operatorname{adj} \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b})}{\det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b})}, \;\; \det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \neq 0 \,. \qquad \text{(C.10)}$$

Obserability

A linear dynamic system is observable, when based on the knowledge of the courses of the control (input) $u(t)$ and output $y(t)$ on the finite interval $t_1 - t_0$ it can be determined the initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

If we know the initial state $\boldsymbol{x}(t_0)$, then we can easily determine the state $\boldsymbol{x}(t)$ for any time $t > t_0$.

Most often we choose $t_0 = 0$.

Because the control (input) $u(t)$ produces some known (forced) response, it is clear that we can choose $u(t) = 0$, ie. We can consider the autonomous linear dynamic system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{Ax}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \,, \qquad \text{(C.11a)}$$

$$y(t) = \boldsymbol{c}^T \boldsymbol{x}(t) \,. \qquad \text{(C.11b)}$$

If we determine for it the initial state $\boldsymbol{x}(0)$, then based on the relation [see (3.43)]

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\, \boldsymbol{x}(0) \qquad \text{(C.12)}$$

we can determine any state $\boldsymbol{x}(t)$ for $t>0$ and from the relation (C.11b) also the corresponding output (3.44)

$$y(t)=\boldsymbol{c}^T\,\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}(0)\,. \tag{C.13}$$

We apply the Sylvester interpolation formula (B.4)

$$\mathrm{e}^{\boldsymbol{A}t}=\sum_{i=0}^{n-1}\alpha_i(t)\boldsymbol{A}^i \tag{C.14}$$

and we obtain

$$y(t)=\boldsymbol{c}^T\left(\sum_{i=0}^{n-1}\alpha_i\boldsymbol{A}^i\right)\boldsymbol{x}(0)=\left(\sum_{i=0}^{n-1}\alpha_i\boldsymbol{c}^T\boldsymbol{A}^i\right)\boldsymbol{x}(0)=$$

$$=[\alpha_0,\alpha_1,\ldots.\alpha_{n-1}]\begin{bmatrix}\boldsymbol{c}^T\\ \boldsymbol{c}^T\boldsymbol{A}\\ \vdots\\ \boldsymbol{c}^T\boldsymbol{A}^{n-1}\end{bmatrix}\boldsymbol{x}(0)\;\Rightarrow$$

$$y(t)=[\alpha_0,\alpha_1,\ldots.\alpha_{n-1}]\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)\boldsymbol{x}(0)\,, \tag{C.15}$$

where

$$\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=\begin{bmatrix}\boldsymbol{c}^T\\ \boldsymbol{c}^T\boldsymbol{A}\\ \vdots\\ \boldsymbol{c}^T\boldsymbol{A}^{n-1}\end{bmatrix}=[\boldsymbol{c},\boldsymbol{A}^T\boldsymbol{c},\ldots,(\boldsymbol{A}^T)^{n-1}\boldsymbol{c}]^T \tag{C.16}$$

is the square observability matrix [see relation (2.51)].

Similarly to the controllability in order to determine on the basis of the relation (C.15) the initial state $\boldsymbol{x}(0)$, the observability matrix (C.16) must be invertible, i.e. it must have the rank n. Since it is the square matrix, its determinant must be zero

$$\operatorname{rank}\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)=n\;\Leftrightarrow\;\det\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)\neq 0\,. \tag{C.17}$$

We can get the same conclusion in other way.

For the autonomous linear dynamic system (C.11) we can write

$$\begin{aligned}
y(0)&=\boldsymbol{c}^T\boldsymbol{x}(0),\\
\dot{y}(0)&=\boldsymbol{c}^T\dot{\boldsymbol{x}}(0)=\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(0),\\
\ddot{y}(0)&=\boldsymbol{c}^T\boldsymbol{A}\dot{\boldsymbol{x}}(0)=\boldsymbol{c}^T\boldsymbol{A}^2\boldsymbol{x}(0),\\
&\vdots\\
y^{(n-1)}(0)&=\boldsymbol{c}^T\boldsymbol{A}^{n-2}\dot{\boldsymbol{x}}(0)=\boldsymbol{c}^T\boldsymbol{A}^{n-1}\boldsymbol{x}(0),
\end{aligned}$$

or

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T \boldsymbol{A}^{n-1} \end{bmatrix} \boldsymbol{x}(0) \; \Rightarrow$$

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \boldsymbol{Q}_{ob}(\boldsymbol{A}, \boldsymbol{c}^T)\boldsymbol{x}(0)\,, \qquad \text{(C.18)}$$

i.e. in order to determine the initial state $\boldsymbol{x}(0)$ from (C.18) for the observability matrix $\boldsymbol{Q}_{ob}(\boldsymbol{A},\boldsymbol{c}^T)$ it must hold (C.17).

For more details see e.g. [Ogata 2010; Mandal 2006; Friedland 2005].

APPENDIX D

Ackerman's formula

For the controllable linear dynamic system with the state model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{D.1}$$

it is necessary to design a state space feedback control represented by a vector $\boldsymbol{k}^T$ which ensures the desired characteristic polynomial of the closed-loop control system in the form

$$N_{kw}(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_w) = s^n + a_{n-1}^w s^{n-1} + \ldots + a_1^w s + a_0^w , \tag{D.2}$$

where

$$\boldsymbol{A}_w = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T . \tag{D.3}$$

In accordance with the Cayley-Hamilton theorem (Appendix B) every square matrix must satisfy its own characteristic equation

$$N_{kw}(\boldsymbol{A}_w) = 0 ,$$

i.e.

$$\boldsymbol{A}_w^n + a_{n-1}^w \boldsymbol{A}_w^{n-1} + \ldots + a_1^w \boldsymbol{A}_w + a_0^w \boldsymbol{I} = \boldsymbol{0} . \tag{D.4}$$

We substitute (D.3) into (D4) and then we modify it. For clarity first we calculate powers of the matrix $\boldsymbol{A}_w$:

$$\boldsymbol{A}_w^2 = (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T)(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T) = \boldsymbol{A}^2 - \boldsymbol{A}\boldsymbol{b}\boldsymbol{k}^T - \boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w , \tag{D.5}$$

$$\begin{aligned} \boldsymbol{A}_w^3 &= (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T)(\boldsymbol{A}^2 - \boldsymbol{A}\boldsymbol{b}\boldsymbol{k}^T - \boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w) = \\ &= \boldsymbol{A}^3 - \boldsymbol{A}^2\boldsymbol{b}\boldsymbol{k}^T - \boldsymbol{A}\boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w - \boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w^2 , \end{aligned} \tag{D.6}$$

$$\vdots$$

$$\boldsymbol{A}_w^n = \boldsymbol{A}^n - \boldsymbol{A}^{n-1}\boldsymbol{b}\boldsymbol{k}^T - \boldsymbol{A}^{n-2}\boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w - \ldots - \boldsymbol{A}\boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w^{n-2} - \boldsymbol{b}\boldsymbol{k}^T \boldsymbol{A}_w^{n-1}. \tag{D.7}$$

Now we substitute (D.3), (D.5) – (D.7) into (D.4), we denote

$$N_{kw}(\boldsymbol{A}) = \boldsymbol{A}^n + a_{n-1}^w \boldsymbol{A}^{n-1} + \ldots + a_1^w \boldsymbol{A} + a_0^w \boldsymbol{I} \tag{D.8}$$

and then we modify the remaining relations so we put $\boldsymbol{b}$, $\boldsymbol{A}\boldsymbol{b}$, $\boldsymbol{A}^2\boldsymbol{b}$ etc. outside brackets and we get

$$\begin{aligned} & N_{kw}(\boldsymbol{A}) - \boldsymbol{b}(a_1^w \boldsymbol{k}^T + a_2^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + a_{n-1}^w \boldsymbol{k}^T \boldsymbol{A}_w^{n-2} + \boldsymbol{k}^T \boldsymbol{A}_w^{n-1}) + \\ & - \boldsymbol{A}\boldsymbol{b}(a_2^w \boldsymbol{k}^T + a_3^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + a_{n-2}^w \boldsymbol{k}^T \boldsymbol{A}_w^{n-3} + a_{n-1}^w \boldsymbol{k}^T \boldsymbol{A}_w^{n-2}) + \ldots \\ & - \boldsymbol{A}^{n-2}\boldsymbol{b}(a_{n-1}^w \boldsymbol{k}^T + \boldsymbol{k}^T \boldsymbol{A}_w) + \boldsymbol{A}^{n-1}\boldsymbol{b}\boldsymbol{k}^T = \boldsymbol{0}. \end{aligned} \tag{D.9}$$

We write down this relation in the matrix form

$$N_{kw}(\boldsymbol{A}) = [\boldsymbol{b}, \boldsymbol{Ab}, \ldots, \boldsymbol{A}^{n-2}\boldsymbol{b}, \boldsymbol{A}^{n-1}\boldsymbol{b}]\begin{bmatrix} a_1^w \boldsymbol{k}^T + a_2^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + \boldsymbol{k}^T \boldsymbol{A}_w^{n-1} \\ a_2^w \boldsymbol{k}^T + a_3^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + a_{n-1}^w \boldsymbol{k}^T \boldsymbol{A}_w^{n-2} \\ \vdots \\ a_{n-1}^w \boldsymbol{k}^T + \boldsymbol{k}^T \boldsymbol{A}_w \\ \boldsymbol{k}^T \end{bmatrix}.$$

The first term on the right side is the controllability matrix

$$\boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{b}, \boldsymbol{Ab}, \ldots, \boldsymbol{A}^{n-2}\boldsymbol{b}, \boldsymbol{A}^{n-1}\boldsymbol{b}] ,$$

which is square and nonsingular [system is controllable, $\det \boldsymbol{Q}_{co}(\boldsymbol{A}, \boldsymbol{b}) \neq 0$], and therefore its inverse exists. Thus we can write

$$\begin{bmatrix} a_1^w \boldsymbol{k}^T + a_2^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + \boldsymbol{k}^T \boldsymbol{A}_w^{n-1} \\ a_2^w \boldsymbol{k}^T + a_3^w \boldsymbol{k}^T \boldsymbol{A}_w + \ldots + a_{n-1}^w \boldsymbol{k}^T \boldsymbol{A}_w^{n-2} \\ \vdots \\ a_{n-1}^w \boldsymbol{k}^T + \boldsymbol{k}^T \boldsymbol{A}_w \\ \boldsymbol{k}^T \end{bmatrix} = \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}, \boldsymbol{b}) N_{kw}(\boldsymbol{A}) .$$

Since we are interested in only vector $\boldsymbol{k}^T$ (last row), so we get

$$\boldsymbol{k}^T = [0, 0, \ldots, 0, 1] \boldsymbol{Q}_{co}^{-1}(\boldsymbol{A}, \boldsymbol{b}) N_{kw}(\boldsymbol{A}) . \qquad \text{(D.10)}$$

APPENDIX E

Desired pole placement

When designing a state space control the pole placement in the left half-plane of the complex plane s is used. The influence of the conjugate pole pair on the step responses of the linear dynamic system of the second order is shown in Fig. E.1.

It is assumed that the linear dynamic system of the second order has the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\dfrac{\xi_0}{\omega_0}s + \omega_0^2} \tag{E.1}$$

or the state model

$$\begin{aligned} &\dot{x}_1(t) = x_2(t), && x_1(0) = 0, \\ &\dot{x}_2(t) = -\omega_0^2 x_1(t) - 2\frac{\xi_0}{\omega_0}x_2(t) + u(t), && x_2(0) = 0, \\ &y(t) = \omega_0^2 x_1(t), \end{aligned} \tag{E.2}$$

where ω_0 is the natural angular frequency (undamped oscillations), ξ_0 – the coefficient of relative damping.

For the assessment of step responses in Fig. E.1 it is appropriate to establish other indices

$$\alpha = \xi_0\omega_0, \;\; \omega = \omega_0\sqrt{1-\xi_0^2}, \;\; \kappa = \frac{y_m - y(\infty)}{y(\infty)}, \tag{E.3}$$

i.e. α – the damping (degree of stability), ω – the angular frequency (damped oscillations), κ – the relative overshoot, y_m – the maximum value of the step response, $y(\infty)$ – the steady-state of the step response.

On the basis of the influence of poles in the left half-plane of the complex plane s on the step response (Fig. E.1) it can be defined so-called **admissible region** for control system poles for the desired damping α_w and relative damping ξ_w accordance with Fig. E.2.

The poles lying the closest to the admissible region boundary are colled the **dominant poles** (sometimes as the dominant poles are thought the ones which are the closest to the imaginary axis).

Further it is assumed that the poles which are located far away from the admissible region boundary have a negligible influence on the control system behavior.

The admissible region boundary in Fig. E.2 is determined by the the following relations

$$\alpha_w \geq (3 \div 5)\frac{1}{t_s}, \tag{E.4}$$

$$\varphi_w \leq \arccos \xi_w . \tag{E.5}$$

where t_s is the settling time, i.e. the time when the output variable $y(t)$ enters in the band with the width 2Δ, i.e. $y(\infty) \pm \Delta$, where the control tolerance $\Delta = \delta y(\infty)$; $\delta = 0.01 \div 0.05$.

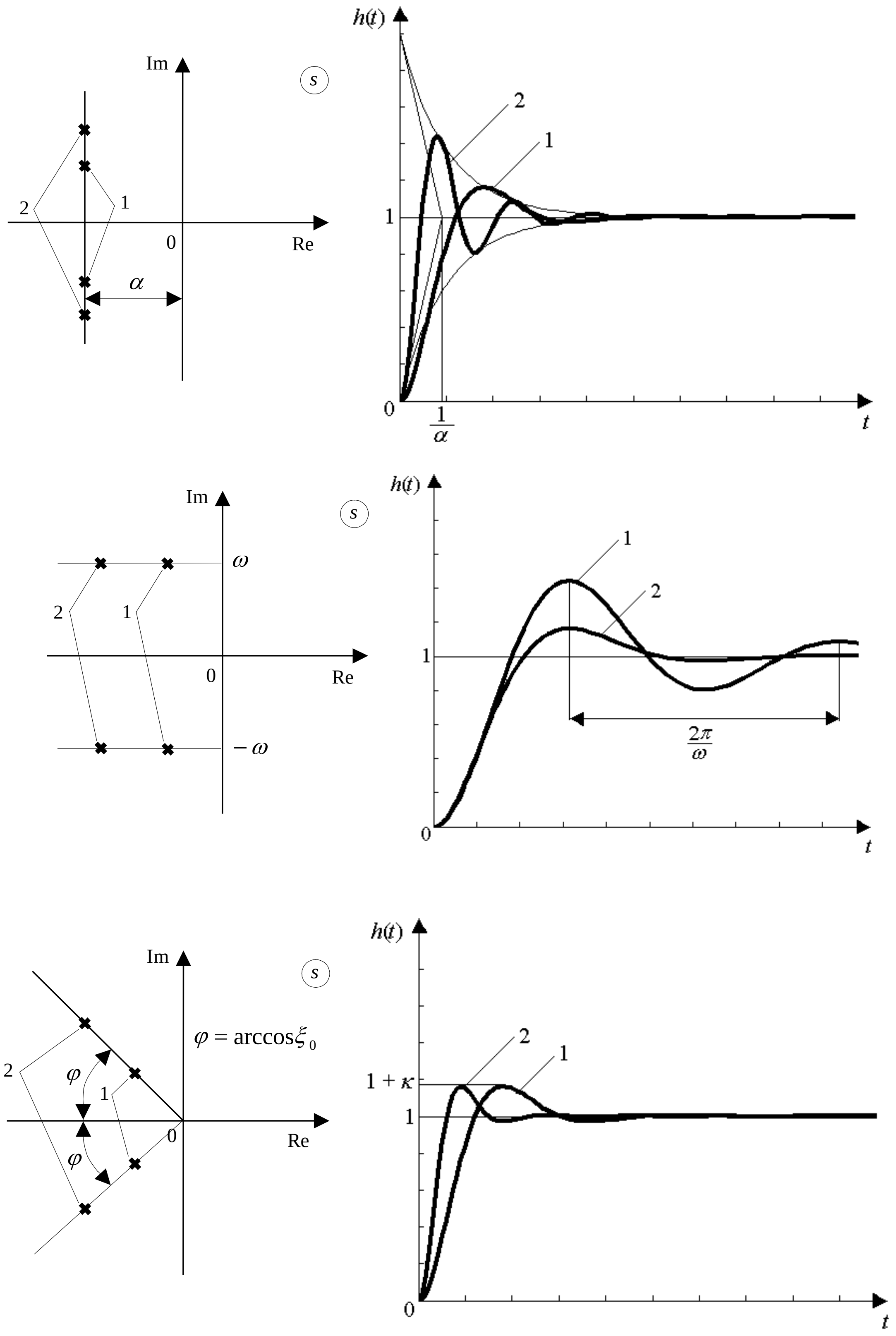


Fig. E.1 Influence of complex conjugate poles of the second order system on its step responses

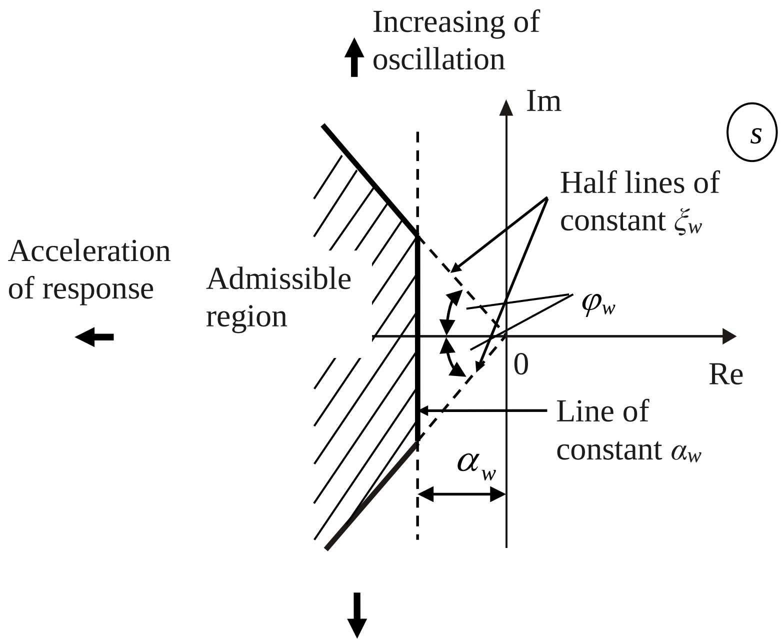


Fig. E.2 Determination of admissible region for control system poles

In the relation (E.4) is the smaller number considered in the case of a single dominant real pole and the greater number in the case of double dominant real pole. The first relation is determined for the control tolerance about 5 %. The second relation (E.5) is based on the assumption of the maximum permissible relative overshoot 25 %, i.e.

$$\kappa \le 0.25 \;\Rightarrow\; \xi_0 \ge 0.404 \;\Rightarrow\; \varphi_w \le 66° \;\; (1.15 \text{ rad}). \tag{E.6}$$

In the design of a state control they are often used the standard binomial forms with the multiple real pole $s_i^w = -a$, $a > 0$ (Fig. E.3):

$$N_{kw}(s) = (s+a)^n , \tag{E.7}$$

$$\begin{aligned} n = 2 \quad & s^2 + 2as + a^2, \\ n = 3 \quad & s^3 + 3as^2 + 3a^2 s + a^3, \\ n = 4 \quad & s^4 + 4as^3 + 6a^2 s^2 + 4a^3 s + a^4, \\ n = 5 \quad & s^5 + 5as^4 + 10a^2 s^3 + 10a^3 s^2 + 5a^4 s + a^5. \end{aligned} \tag{E.8}$$

The integral criterion ITAE is very popular

$$I_{ITAE} = \int_0^\infty t\left|e(t)\right| \mathrm{d}\, t \to \min . \tag{E.9}$$

The integral criterion ITAE I_{ITAE} (ITAE = **I**ntegral of **T**ime multiplied by **A**bsolute **E**rror) includes time and control error, and therefore when it is minimized then both the absolute control area and the settling time t_s are simultaneously minimized. The integral criterion ITAE is very popular although its value can be determined in most cases only by simulation.

Original coefficients of a desired characteristic polynomial $N_w(s)$ which are given e.g. in [Graham, Lathrop 1953] were obtained on the basis of an analog simulation and later they were precised by a digital simulation [Cao 2008]. New standard forms give a substantially smaller value of the integral criterion ITAE (E.9) primarily for higher degrees of characteristic polynomials.

New coefficients of characteristic polynomials for the criterion ITAE (Fig. E.4):

$$\begin{aligned} n&=2 \quad s^2+1{,}505as+a^2, \\ n&=3 \quad s^3+1{,}783as^2+2{,}172a^2s+a^3, \\ n&=4 \quad s^4+1{,}953as^3+3{,}347a^2s^2+2{,}648a^3s+a^4, \\ n&=5 \quad s^5+2{,}068as^4+4{,}499a^2s^3+4{,}675a^3s^2+3{,}257a^4s+a^5. \end{aligned} \tag{E.10}$$

The constant a in the relations (E7), (E.8) and (E.10) expresses the time scale. Its choice adapts the standard form of the characteristic polynomial to the real system.

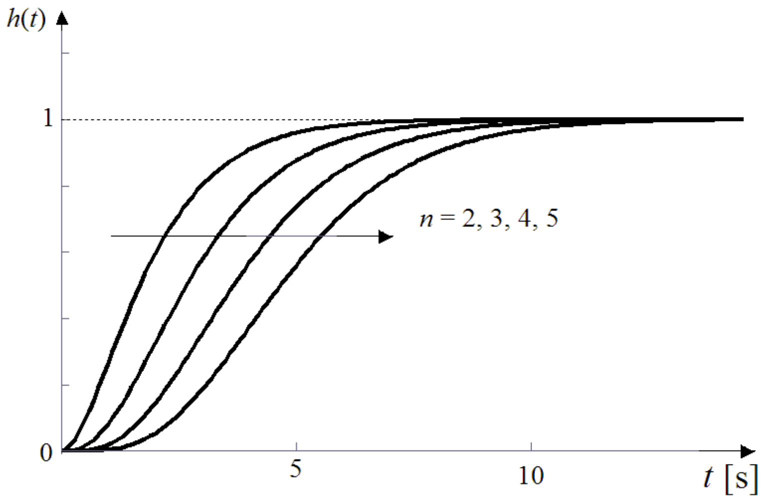


Fig. E.3 Step responses for binomial standard forms (E.8) for a = 1

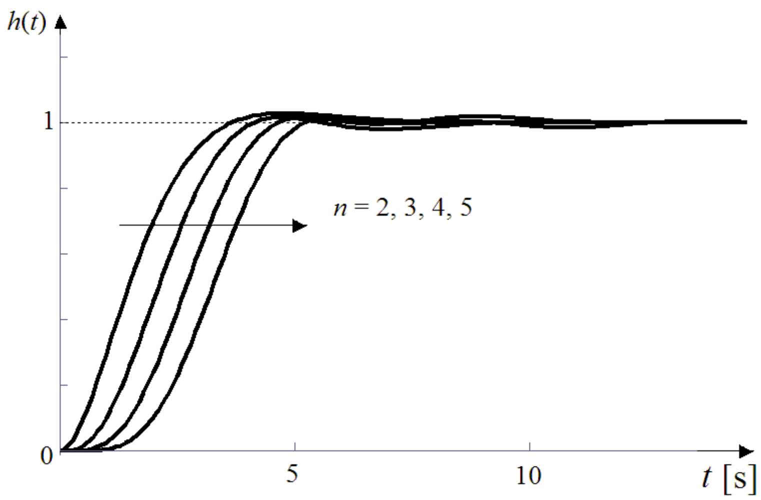


Fig. E.4 Step responses for ITAE standard forms (E.10) for a = 1

Fig. E.3 and E.4 show the step responses for binomial and ITAE standard forms for $a = 1$.

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Authors:	Prof. Ing. Miluše Vítečková, CSc. Prof. Ing. Antonín Víteček, CSc., Dr.h.c.
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